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# Fundamentals of the double – decomposition concept for turbulent transport in permeable media

## Grundlagen des Double–Decomposition-Konzepts für turbulenten Transport in permeablen Medien

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Environmental impact analyses as well as engineering equipment design can both benefit from reliable modeling of turbulent flow in porous media. A number of natural and engineering systems can be characterized by a permeable structure through which a working fluid permeates. Turbulence models proposed for such flows depend on the order of application of time and volume average operators. Two methodologies, following the two orders of integration, lead to different governing equations for the statistical quantities. This paper reviews recently published methodologies to mathematically characterize turbulent transport in porous media. A new concept, called double-decomposition, is here discussed and models for turbulent transport in porous media are classified in terms of the order of application of the time and volume averaging operators, among other peculiarities.

Within this paper Instantaneous Local Transport Equations are reviewed for clear flow before Time and Volume Averaging Procedures are applied to them. The Double-Decomposition Concept is presented and thoroughly discussed prior the derivation of macroscopic governing equations. Equations for Turbulent Transport follow, showing detailed derivation for the mean and turbulent field quantities.

Key words: Turbulent flow; Double-Decomposition; Porous Structures, Permeable media, Mathematical Modeling.

Umweltverträglichkeitsanalysen sowie das Design von technischer Ausrüstung profitieren von der zuverlässigen Modellierung turbulenter Strömung in porösen Medien. Viele natürliche und technische Systeme können mittels einer permeablen Struktur beschrieben werden, welche von einem Arbeitsmedium durchdrungen wird. Vorhandene Turbulenzmodelle solcher Strukturen sind abhängig von der Größenordnung der Zeit und den durchschnittlichen Volumenoperatoren. Zwei Methoden - entsprechend den beiden Integrationsgraden, führen zu verschiedenen problembeschreibenden Differentialgleichungen zur Beschreibung der statistischen Größen. Dieser Artikel diskutiert kürzlich veröffentlichte Methoden für die mathematische Charakterisierung von turbulenter Strömung in porösen Medien. Ein neues Konzept - als „double-decomposition“ bezeichnet - wird diskutiert und Modelle des turbulenten Transports in porösen Medien werden klassifiziert in Abhängigkeit der Größenordnung der Zeit und von durchschnittlichen Volumenoperatoren zusammen mit weiteren Charakteristika.

Im Rahmen dieses Artikels werden unmittelbare lokale Transportgleichungen der reinen Strömung beurteilt, bevor zeit- und volumenmittelnde Verfahren angewendet werden. Das Double-Decomposition-Konzept wird vorgestellt und vor der Ableitung makroskopischer Gleichungen ausführlich diskutiert. Des Weiteren folgen Gleichungen für den turbulenten Transport mit detaillierter Ableitung der mittleren und turbulenten Feldgrößen.

Schlüsselworte: Turbulente Strömung; Double-Decomposition; poröse Strukturen; permeable Medien; mathematische Modellierung.

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## 1 Introduction

It is well established in the literature that modelling of macroscopic transport for incompressible flows in porous media can be based on the volume-average methodology [1] for either heat [2] or mass transfer [3-6]. If the fluid phase properties fluctuate with time, in addition to presenting spatial deviations, there are two possible methodologies to follow in order to obtain macroscopic equations: (a) application of time-average operator followed by volume-averaging [7-11], or (b) use of volume-averaging before time-averaging is applied [12-15]. In fact, these two sets of macroscopic transport equations are equivalent when examined under the recently established double decomposition concept [16-20]. Recent reviews on the topic of turbulence in permeable media can be found in References [21-23]. Advances on the general area of porous media are found in recently published books devoted to the subject [24-26].

The double-decomposition idea was initially developed for the flow variables in porous media and has been extended to non-buoyant heat transfer [27-28], buoyant flows [29-31], mass transfer [32], non-equilibrium heat transfer [33], double-diffusive transport [34], and hybrid media (clear/porous domains) [35]. The problem of treating macroscopic interfaces bounding finite porous media, considering a diffusion-jump condition for the mean [36, 37] and turbulence fields [38], has also been investigated under the concept first proposed by Ref. [17]. A general classification of all proposed models for turbulent flow and heat transfer in porous media has been recently published [39]. Here, a systematic review of this new concept is presented.

## 2 Transport equations

The steady-state local or microscopic instantaneous transport equations for an incompressible fluid with constant properties are given by:

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$$\nabla \cdot \mathbf{u} = 0 \quad (1)$$

$$\rho \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} \quad (2)$$

where  $\mathbf{u}$  is the velocity vector,  $\rho$  is the density,  $p$  is the pressure,  $\mu$  is the fluid viscosity and  $\mathbf{g}$  is the gravity acceleration vector.

As mentioned, there are, in principle, two ways that one can follow in order to treat turbulent flow in porous media. The first method applies a time average operator to the governing Eqs. (1)–(2) before the volume average procedure is applied. In the second approach, the order of application of the two average operators is reversed. Both techniques aim at derivation of suitable macroscopic transport equations.

Volume averaging in a porous medium, described in detail in Refs. [40–42], makes use of the concept of a Representative Elementary Volume (REV) over which local equations are integrated. In a similar fashion, statistical analysis of turbulent flow leads to time mean properties. Transport equations for statistical values are considered in lieu of instantaneous information on the flow.

## 2.1 Time and Volume Averaging Procedures

Traditional analyses of turbulence are based on statistical quantities, which are obtained by applying time-averaging to the flow governing equations. As such, the time average of a general quantity  $\varphi$  is defined as follows:

$$\bar{\varphi} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \varphi dt \quad (3)$$

where the time interval  $\Delta t$  is small compared to the fluctuations of the average value,  $\bar{\varphi}$ , but large enough to capture turbulent fluctuations of  $\varphi$ . Time decomposition can then be written as follows:

$$\varphi = \bar{\varphi} + \varphi' \quad (4)$$

with  $\bar{\varphi}' = 0$ , where  $\varphi'$  is the time fluctuation of  $\varphi$  around its average value  $\bar{\varphi}$ .

The volume average of a general property  $\varphi$  taken over a REV in a porous medium can be written as [40],

$$\langle \varphi \rangle^v = \frac{1}{\Delta V} \int_{\Delta V} \varphi dV \quad (5)$$

The value  $\langle \varphi \rangle^v$  is defined for any point  $x$  surrounded by a REV of size  $\Delta V$ . This average is related to the intrinsic average for the fluid phase as follows:

$$\langle \varphi \rangle^v = \phi \langle \varphi \rangle^i \quad (6)$$

where  $\phi = \Delta V_f / \Delta V$  is the local medium porosity and  $\Delta V_f$  is the volume occupied by the fluid in a REV. Furthermore, we can write,

$$\varphi = \langle \varphi \rangle^i + {}^i\varphi \quad (7)$$

with  $\langle {}^i\varphi \rangle^i = 0$ . In Eq. (7),  ${}^i\varphi$  is the spatial deviation of  $\varphi$  with respect to the intrinsic average  $\langle \varphi \rangle^i$ .

For deriving the flow governing equations, it is necessary to know the relationship between the volumetric average of derivatives and the derivatives of the volumetric average. These relationships are presented in a number of works, e.g. Ref [1, 41], being known as the Theorem of Local Volumetric Average. They are written as follows:

$$\langle \nabla \varphi \rangle^v = \nabla (\phi \langle \varphi \rangle^i) + \frac{1}{\Delta V} \int_{A_i} \mathbf{n} \varphi dS \quad (8)$$

$$\langle \nabla \cdot \varphi \rangle^v = \nabla \cdot (\phi \langle \varphi \rangle^i) + \frac{1}{\Delta V} \int_{A_i} \mathbf{n} \cdot \varphi dS \quad (9)$$

$$\left\langle \frac{\partial \varphi}{\partial t} \right\rangle^v = \frac{\partial}{\partial t} (\phi \langle \varphi \rangle^i) - \frac{1}{\Delta V} \int_{A_i} \mathbf{n} \cdot (\mathbf{u}_i \varphi) dS \quad (10)$$

where  $A_i$ ,  $\mathbf{u}_i$  and  $\mathbf{n}$  are the interfacial area, the velocity of phase  $f$  and the unit vector normal to  $A_i$ , respectively.

The area  $A_i$  should not be confused with the surface area surrounding volume  $\Delta V$ . To the interested reader, mathematical details and proof of the Theorem of Local Volumetric Average can be found in Refs. [1, 40–42]. For single-phase flow, phase  $f$  is the fluid itself and  $\mathbf{u}_i=0$  if the porous substrate is assumed to be fixed. In developing Eqs. (8)–(10), the only restriction applied is the independence of  $\Delta V$  in relation to time and space. If the medium is further assumed to be rigid, then  $\Delta V_f$  is dependent only on space and not time-dependent [42].

## 2.2 Time Averaged Transport Equations

In order to apply the time average operator to equations (1), (2) and (8), we consider:

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \quad p = \bar{p} + p' \quad (12)$$

Substituting expression (12) into Eqs. (1), (2) and (8), respectively, we obtain after considering constant flow properties,

$$\nabla \cdot \bar{\mathbf{u}} = 0 \quad (13)$$

$$\rho \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -(\nabla \bar{p}) + \mu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot (-\rho \overline{\mathbf{u}'\mathbf{u}'}) \quad (14)$$

For a clear fluid, the use of the eddy-diffusivity concept for expressing the stress-rate of strain relationship for the Reynolds stress appearing in Eq. (14) gives,

$$-\rho \overline{\mathbf{u}'\mathbf{u}'} = \mu_t 2\bar{\mathbf{D}} - \frac{2}{3} \rho k \mathbf{I} \quad (15)$$

where  $\bar{\mathbf{D}}$  is the mean deformation tensor,  $k$  is the turbulent kinetic energy per unit mass,  $\mu_t$  is the turbulent viscosity and  $\mathbf{I}$  is the unity tensor.

The transport equation for the turbulent kinetic energy is obtained by multiplying first the difference between the instantaneous and the time-averaged momentum equations by  $\mathbf{u}'$ . Thus, applying further the time average operator to the resulting product, we obtain,

$$\rho \nabla \cdot (\bar{\mathbf{u}}k) = -\rho \nabla \cdot \left[ \overline{\mathbf{u}' \left( \frac{p'}{\rho} + q \right)} \right] + \mu \nabla^2 k + P_k - \rho \varepsilon \quad (16)$$

where  $P_k = -\rho \overline{\mathbf{u}'\mathbf{u}'} : \nabla \overline{\mathbf{u}}$  is the generation rate of  $k$  due to gradients of the mean velocity and  $q = \mathbf{u}' \cdot \mathbf{u}'/2$ .

### 3 The Double-decomposition concept

The double decomposition idea, herein used for obtaining macroscopic equations, has been detailed in Refs. [16-20]. Here, a general overview is presented. Further, the resulting equations using this concept for the flow [17] and non-buoyant thermal fields [27, 28] are already available in the literature and because of this they are not reviewed here in great detail. As already mentioned, extensions of the double-decomposition methodology to buoyant flows [29-30], to mass transport [32], and to double-diffusive convection [34], have also been presented in the open literature.

Basically, for porous media analysis, a macroscopic form of the governing equations is obtained by taking the volumetric average of the entire equation set. In that development, the porous medium is considered to be rigid and saturated by an incompressible fluid.

**Basic Relationships.** From the work in References [16, 27], one can write for any flow property  $\varphi$  combining decompositions (7) and (4),

$$\langle \varphi \rangle^i = \langle \bar{\varphi} \rangle^i + \langle \varphi' \rangle^i; \quad \bar{\varphi} = \langle \bar{\varphi} \rangle^i + {}^i\bar{\varphi};$$

$${}^i\varphi = {}^i\bar{\varphi} + {}^i\varphi'; \quad \varphi' = \langle \varphi' \rangle^i + {}^i\varphi' \quad (17)$$

or further

$$\varphi' = \langle \varphi' \rangle^i + {}^i\varphi' = \langle \varphi' \rangle^i + {}^i\varphi' \quad (18)$$

where  ${}^i\varphi'$  can be understood as either the time fluctuation of the spatial deviation or the spatial deviation of the time fluctuation. After some manipulation, we can prove that [17],

$$\overline{\langle \varphi \rangle^v} = \langle \bar{\varphi} \rangle^v; \quad \text{or} \quad \overline{\langle \varphi' \rangle^i} = \langle \bar{\varphi} \rangle^i \quad (19)$$

i.e. the time and volume averages commute. Also,

$${}^i\bar{\varphi} = \overline{{}^i\varphi}; \quad \langle \varphi' \rangle^i = \langle \varphi' \rangle^i \quad (20)$$

or say,

$$\langle \varphi \rangle^i = \frac{1}{\Delta V_f} \int_{\Delta V_f} \varphi dV = \frac{1}{\Delta V_f} \int_{\Delta V_f} (\bar{\varphi} + \varphi') dV \quad (21)$$

$$= \langle \bar{\varphi} \rangle^i + \langle \varphi' \rangle^i$$

$${}^i\varphi = {}^i\bar{\varphi} + {}^i\varphi' = (\overline{{}^i\varphi}) + ({}^i\varphi)' \quad (22)$$

so that,

$$\varphi' = \langle \varphi' \rangle^i + {}^i\varphi' \quad \text{where}$$

$${}^i\varphi' = \varphi' - \langle \varphi' \rangle^i \quad \text{and also} \quad ({}^i\varphi)' = {}^i\varphi - \overline{{}^i\varphi} \quad (23)$$

Finally, we can have a full variable decomposition as follows:

$$\varphi = \underbrace{\langle \bar{\varphi} \rangle^i}_{\langle \varphi \rangle^i} + \underbrace{\langle \varphi' \rangle^i}_{\langle \varphi' \rangle^i} + {}^i\bar{\varphi} + {}^i\varphi' = \overline{\langle \varphi \rangle^i} + \langle \varphi' \rangle^i + \overline{{}^i\varphi} + ({}^i\varphi)' \quad (24)$$

or further,

$$\varphi = \underbrace{\langle \bar{\varphi} \rangle^i}_{\langle \varphi \rangle^i} + \underbrace{\langle \varphi' \rangle^i}_{\langle \varphi' \rangle^i} + \overline{{}^i\varphi} + ({}^i\varphi)' = \overline{\langle \varphi \rangle^i} + \overline{{}^i\varphi} + \underbrace{\langle \varphi' \rangle^i + ({}^i\varphi)'}_{\langle \varphi' \rangle^i} \quad (25)$$

Equation (24) comprises the double decomposition concept. The significance of the four terms in expression (25) can be reviewed as: (a)  $\overline{\langle \varphi \rangle^i}$ , is the intrinsic average of the time mean value of  $\varphi$ , i.e. we compute first the time averaged values of all points composing the REV, and then we find their volumetric mean to obtain  $\overline{\langle \varphi \rangle^i}$ . Instead, we could also consider a certain point  $x$  surrounded by the REV, according to Eq. (5) and (6), and take the volumetric average at different time steps. Thus, we calculate the average over such different values of  $\langle \varphi \rangle^i$  in time. We get then  $\overline{\langle \varphi \rangle^i}$  and, according to expression (19),  $\overline{\langle \varphi \rangle^i} = \langle \bar{\varphi} \rangle^i$ , i.e. the volumetric and time average commute. (b) If we now take the volume average of all fluctuating components of  $\varphi$ , which compose the REV, we end up with  $\langle \varphi' \rangle^i$ . Instead, with the volumetric average around point  $x$  taken at different time steps we can determine the difference between the instantaneous and a time averaged value. This will be  $\langle \varphi' \rangle^i$  that, according to expression (20), equals  $\langle \varphi' \rangle^i$ . Further, on performing first a time averaging operation over all points that contribute with their local values to the REV, we get a distribution of  $\bar{\varphi}$  within this volume. If now we calculate the intrinsic average of this distribution of  $\bar{\varphi}$ , we get  $\overline{\langle \bar{\varphi} \rangle^i}$ . The difference or deviation between these two values is  ${}^i\bar{\varphi}$ . Now, using the same space decomposition approach, we can find for any instant of time  $t$  the deviation  ${}^i\varphi$ . This value also fluctuates with time, and as such a time mean can be calculated as  $\overline{{}^i\varphi}$ . Again the use of expression (20) gives  $\overline{{}^i\varphi} = \langle \bar{\varphi} \rangle^i$ . Finally, it is interesting to note the meaning of the last term on each side of Eq. (25). The first term,  $({}^i\varphi)'$ , is the time fluctuation of the spatial component whereas  ${}^i\varphi'$  means the spatial component of the time varying term. If, however, one makes use of relationships (19) and (20) to simplify expression (25), we finally conclude,

$${}^i\varphi' = ({}^i\varphi)' \quad (26)$$

and, for simplicity of notation, we can drop the parentheses and write both superscripts at the same level in the format:  ${}^i\varphi'$ .

Also,  $\langle ({}^i\varphi)' \rangle^i = \overline{{}^i\varphi'} = 0$ .

The basic advantage of the double decomposition concept is to serve as a mathematical framework for analysis of flows where within the fluid phase there is enough room for turbulence to be established. As such, the double-decomposition methodology would be useful in situations where a solid phase is present in the domain under analysis so that a macroscopic view is appropriate. At the same time, properties in the fluid phase are subjected to the turbulent regime, and a statistical approach is appropriate. Examples of possible applications of such methodology can be found in engineering systems such as heat exchangers, porous combustors, nuclear reactor cores, etc. Natural systems include atmospheric boundary layer over forests and crops.

## 4 Momentum transport

### Mean Flow

The development to follow assumes single-phase flow in a saturated, rigid porous medium ( $\Delta V_f$  independent of time) for which, in accordance with expression (19), time average operation on the variable  $\varphi$  commutes with the space average. Application of the double decomposition idea in Eq. (25) to the inertia term in the momentum equation leads to four different terms. Not all of these terms are considered in the same analysis in the literature.

**Continuity.** The microscopic continuity equation for an incompressible fluid flowing in a clear (non-porous) domain was given by equation (1). Using the double decomposition idea embodied in expression (25) results,

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i + \bar{\mathbf{u}} + \mathbf{u}') = 0 \quad (27)$$

On applying both volume and time-average operators in either order gives,

$$\nabla \cdot (\phi \langle \bar{\mathbf{u}} \rangle^i) = 0 \quad (28)$$

As such, for the continuity equation the averaging order is immaterial.

**Momentum - one average operator.** The transient form of the microscopic momentum Eq. (2) for a fluid with constant properties is given by the Navier-Stokes equation as follows:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) \right] = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} \quad (29)$$

Its time-average, using  $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ , gives

$$\rho \left[ \frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) \right] = -\nabla \bar{p} + \mu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot (-\rho \overline{\mathbf{u}'\mathbf{u}'}) + \rho \mathbf{g} \quad (30)$$

where the stresses,  $-\rho \overline{\mathbf{u}'\mathbf{u}'}$ , are the well-known Reynolds stresses. On the other hand, the volumetric average of Eq. (29) using the **Theorem of Local Volumetric Average**, Eqs. (8)-(10), results in the following:

$$\begin{aligned} \rho \left[ \frac{\partial}{\partial t} (\phi \langle \mathbf{u} \rangle^i) + \nabla \cdot [\phi \langle \mathbf{u}\mathbf{u} \rangle^i] \right] \\ = -\nabla (\phi \langle p \rangle^i) + \mu \nabla^2 (\phi \langle \mathbf{u} \rangle^i) + \phi \rho \mathbf{g} + \mathbf{R} \end{aligned} \quad (31)$$

where

$$\mathbf{R} = \frac{\mu}{\Delta V} \int_{A_i} \mathbf{n} \cdot (\nabla \mathbf{u}) dS - \frac{1}{\Delta V} \int_{A_i} \mathbf{n} p dS \quad (32)$$

represents the total drag force per unit volume due to the presence of the porous matrix, being composed by both viscous drag and form (pressure) drags. Further, using spatial decomposition to write  $\mathbf{u} = \langle \mathbf{u} \rangle^i + \mathbf{u}'$  in the inertia term we obtain the following:

$$\begin{aligned} \rho \left[ \frac{\partial}{\partial t} (\phi \langle \mathbf{u} \rangle^i) + \nabla \cdot [\phi \langle \mathbf{u} \rangle^i \langle \mathbf{u} \rangle^i] \right] = -\nabla (\phi \langle p \rangle^i) \\ + \mu \nabla^2 (\phi \langle \mathbf{u} \rangle^i) - \nabla \cdot [\phi \langle \mathbf{u}'\mathbf{u}' \rangle^i] + \phi \rho \mathbf{g} + \mathbf{R} \end{aligned} \quad (33)$$

Reference [2] pointed out that the third term on the right hand side represents the hydrodynamic dispersion due to spatial deviations. Note that equation (33) models typical porous media flow for  $Re_p < 150-200$ . When extending the analysis to turbulent flow, time varying quantities have to be considered.

**Momentum equation – two average operators.** The set of Eq. (30) and (33) are used when treating turbulent flow in clear fluid, or low  $Re_p$  porous media flow, respectively. In each one of those equations only one averaging operator was applied, either time or volume, respectively. In this work, an investigation on the use of both operators is now conducted with the objective of modeling turbulent flow in porous media.

The volume average of Eq. (30) gives for the time mean flow in a porous medium,

$$\begin{aligned} \rho \left[ \frac{\partial}{\partial t} (\phi \langle \bar{\mathbf{u}} \rangle^i) + \nabla \cdot (\phi \langle \bar{\mathbf{u}}\bar{\mathbf{u}} \rangle^i) \right] = -(\phi \langle \bar{p} \rangle^i) \\ + \mu \nabla^2 (\phi \langle \bar{\mathbf{u}} \rangle^i) + \nabla \cdot (-\rho \phi \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i) + \phi \rho \mathbf{g} + \bar{\mathbf{R}}, \end{aligned} \quad (34)$$

where,

$$\bar{\mathbf{R}} = \frac{\mu}{\Delta V} \int_{A_i} \mathbf{n} \cdot (\nabla \bar{\mathbf{u}}) dS - \frac{1}{\Delta V} \int_{A_i} \mathbf{n} \bar{p} dS \quad (35)$$

is the time-averaged total drag force per unit volume, due to solid particles, composed by both viscous and form (pressure) drags.

Likewise, applying now the time average operation to Eq. (31), we obtain,

$$\begin{aligned} \rho \left[ \frac{\partial}{\partial t} (\overline{\phi \langle \mathbf{u} + \mathbf{u}' \rangle^i}) + \nabla \cdot (\overline{\phi \langle (\bar{\mathbf{u}} + \mathbf{u}')(\bar{\mathbf{u}} + \mathbf{u}') \rangle^i}) \right] \\ = -\nabla (\overline{\phi \langle \bar{p} + p' \rangle^i}) + \mu \nabla^2 (\overline{\phi \langle \bar{\mathbf{u}} + \mathbf{u}' \rangle^i}) + \phi \rho \mathbf{g} + \bar{\mathbf{R}} \end{aligned} \quad (36)$$

Dropping terms containing only one fluctuating quantity results in,

$$\begin{aligned} \rho \left[ \frac{\partial}{\partial t} (\phi \langle \bar{\mathbf{u}} \rangle^i) + \nabla \cdot (\phi \langle \bar{\mathbf{u}}\bar{\mathbf{u}} \rangle^i) \right] = -\nabla (\phi \langle \bar{p} \rangle^i) \\ + \mu \nabla^2 (\phi \langle \bar{\mathbf{u}} \rangle^i) + \nabla \cdot (-\rho \phi \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i) + \phi \rho \mathbf{g} + \bar{\mathbf{R}} \end{aligned} \quad (37)$$

where

$$\begin{aligned} \bar{\mathbf{R}} = \frac{\mu}{\Delta V} \int_{A_i} \mathbf{n} \cdot [\nabla (\bar{\mathbf{u}} + \mathbf{u}')] dS - \frac{1}{\Delta V} \int_{A_i} \mathbf{n} (\bar{p} + p') dS \\ = \frac{\mu}{\Delta V} \int_{A_i} \mathbf{n} \cdot (\nabla \bar{\mathbf{u}}) dS - \frac{1}{\Delta V} \int_{A_i} \mathbf{n} \bar{p} dS \end{aligned} \quad (38)$$

Comparing Eq. (34) and (37), we can see that also for the momentum equation the order of the application of both averaging operators is immaterial.

It is interesting to emphasize that both views in the literature use the same final form for the momentum equation. The term  $\bar{\mathbf{R}}$  is modeled by the Darcy-Forchheimer (Dupuit) expression after either order of application of the average opera-

tors. Since both orders of integration lead to the same equation, namely expression (35) or (38), there would be no reason for modeling them in a different form. Had the outcome of both integration processes been distinct, the use of a different model for each case would have been consistent. In fact, it has been pointed out by Ref [16] that the major difference between those two paths lies in the definition of a suitable turbulent kinetic energy for the flow. Accordingly, the source of controversies comes from the inertia term, as seen below.

**Inertia term - space and time (double) decomposition:**

Applying the double decomposition idea seen before for velocity (Eq. (25) ), to the inertia term of Eq. (29) will lead to different sets of terms. In the literature, not all of them are used in the same analysis.

Starting with time decomposition and applying both average operators, see Eq. (34), gives,

$$\begin{aligned} \nabla \cdot \overline{(\phi \langle \mathbf{u} \mathbf{u} \rangle^i)} &= \nabla \cdot \overline{(\phi \langle (\bar{\mathbf{u}} + \mathbf{u}') (\bar{\mathbf{u}} + \mathbf{u}') \rangle^i)} \\ &= \nabla \cdot [\phi \langle \overline{\langle \mathbf{u} \mathbf{u} \rangle^i} \rangle + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \end{aligned} \quad (39)$$

using spatial decomposition to write  $\bar{\mathbf{u}} = \langle \bar{\mathbf{u}} \rangle^i + \bar{\mathbf{u}}$  we obtain,

$$\begin{aligned} \nabla \cdot [\phi \langle \overline{\langle \mathbf{u} \mathbf{u} \rangle^i} \rangle + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] &= \nabla \cdot \{ \phi [ \langle \overline{\langle (\bar{\mathbf{u}})^i + \bar{\mathbf{u}} \rangle (\langle \bar{\mathbf{u}} \rangle^i + \bar{\mathbf{u}})} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \\ &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \end{aligned} \quad (40)$$

Now, applying Eq. (18) to write  $\mathbf{u}' = \langle \mathbf{u}' \rangle^i + \mathbf{u}'$ , and substituting into expression (40) gives,

$$\begin{aligned} \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\langle \langle \mathbf{u}' \rangle^i + \mathbf{u}' \rangle (\langle \mathbf{u}' \rangle^i + \mathbf{u}') \rangle^i} ] \} \\ &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\langle \langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} + \langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i + \langle \mathbf{u}' \mathbf{u}' \rangle^i + \langle \mathbf{u}' \mathbf{u}' \rangle^i} ] \} \\ &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\langle \langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} + \langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i + \langle \mathbf{u}' \mathbf{u}' \rangle^i} ] \} \end{aligned} \quad (41)$$

The fourth and fifth terms on the right hand side contains only one space varying quantity and will vanish under the application of volume integration. Eq. (41) will then be reduced to,

$$\nabla \cdot \overline{(\phi \langle \mathbf{u} \mathbf{u} \rangle^i)} = \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \quad (42)$$

Using the equivalence (19)-(20), Eq. (42) can be further rewritten as follows,

$$\nabla \cdot \overline{(\phi \langle \mathbf{u} \mathbf{u} \rangle^i)} = \nabla \cdot \{ \phi [ \langle \overline{\langle \mathbf{u} \rangle^i \langle \mathbf{u} \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \quad (43)$$

with an interpretation of the terms in Eq. (42) given later.

Another route to follow to reach the same results is to start out with the application of the space decomposition in the inertia term, as usually done in classical mathematical treatment of porous media flow analysis. Then we obtain,

$$\begin{aligned} \nabla \cdot \overline{(\phi \langle \mathbf{u} \mathbf{u} \rangle^i)} &= \nabla \cdot \overline{(\phi \langle (\langle \mathbf{u} \rangle^i + \mathbf{u}') (\langle \mathbf{u} \rangle^i + \mathbf{u}') \rangle^i)} \\ &= \nabla \cdot [\phi \langle \overline{\langle \mathbf{u} \rangle^i \langle \mathbf{u} \rangle^i} \rangle + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \end{aligned} \quad (44)$$

and on time averaging the r.h.s, using Eq. (21) to express  $\langle \mathbf{u} \rangle^i = \langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i$ , becomes,

$$\begin{aligned} \nabla \cdot \overline{(\phi \langle \langle \mathbf{u} \rangle^i \langle \mathbf{u} \rangle^i + \langle \mathbf{u}' \mathbf{u}' \rangle^i)} &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i} \rangle (\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i) + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \\ &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \end{aligned} \quad (45)$$

With the help of Eq. (22) one can write  $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$  which, inserted into expression (45), gives,

$$\begin{aligned} \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} + \langle \overline{\langle \langle \bar{\mathbf{u}} + \mathbf{u}' \rangle^i \langle \bar{\mathbf{u}} + \mathbf{u}' \rangle^i} \rangle] \} \\ &= \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} + \langle \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} + \langle \overline{\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i} \rangle \langle \overline{\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i} \rangle + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} \end{aligned} \quad (46)$$

Application of the time average operator to the fourth and fifth terms on the right hand side of Eq. (46), containing only one fluctuating component, vanishes it. In addition, remembering that with expression (20) the equivalences  $\langle \bar{\mathbf{u}} \rangle^i = \langle \bar{\mathbf{u}} \rangle^i$  and  $\langle \mathbf{u}' \rangle^i = \langle \mathbf{u}' \rangle^i$  are valid, and that with expression (19) we can write  $\langle \mathbf{u}' \rangle^i = \langle \bar{\mathbf{u}} \rangle^i$ , we obtain the following alternative form for Eq. (46),

$$\begin{aligned} \nabla \cdot \overline{(\phi \langle \langle \mathbf{u} \rangle^i \langle \mathbf{u} \rangle^i + \langle \mathbf{u}' \mathbf{u}' \rangle^i)} &= \nabla \cdot \phi [ \underbrace{\langle \overline{\langle \bar{\mathbf{u}} \rangle^i \langle \bar{\mathbf{u}} \rangle^i} \rangle}_I + \underbrace{\langle \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} \rangle}_II + \underbrace{\langle \overline{\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i} \rangle}_III + \underbrace{\langle \overline{\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i} \rangle}_IV ] \end{aligned} \quad (47)$$

which is the same result as expression (42).

The physical significance of all four terms on the right hand side of (47) can be discussed as:

- I. **Convective** term of macroscopic mean velocity,
- II. **Turbulent (Reynolds) Stresses** divided by the density  $\rho$  due to the fluctuating component of the macroscopic velocity,
- III. **Dispersion** associated with spatial fluctuations of microscopic time mean velocity. Note that this term is also present in the laminar flow, or say, when the pore-based Reynolds number,  $Re_p$ , is less than 150, and
- IV. **Turbulent dispersion** in a porous medium due to both time and spatial fluctuations of the microscopic velocity.

## 5 Fluctuating velocity

The starting point for an equation for the flow turbulent kinetic energy is an equation for the microscopic velocity fluctuation  $\mathbf{u}'$ . Such a relationship can be written, after subtracting the equation for the mean velocity  $\bar{\mathbf{u}}$  from the instantaneous momentum equation, resulting in the following:

$$\rho \left\{ \frac{\partial \mathbf{u}'}{\partial t} + \nabla \cdot [\bar{\mathbf{u}} \mathbf{u}' + \mathbf{u}' \bar{\mathbf{u}} + \mathbf{u}' \mathbf{u}' - \overline{\mathbf{u}' \mathbf{u}'}] \right\} = -\nabla p' + \mu \nabla^2 \mathbf{u}' \quad (48)$$

Now, the volumetric average of Eq. (48) using the Theorem of Local Volumetric Average, gives,

$$\begin{aligned} \rho \frac{\partial}{\partial t} (\phi \langle \mathbf{u}' \rangle^i) + \rho \nabla \cdot \{ \phi [ \langle \overline{\langle \bar{\mathbf{u}} \rangle^i} \rangle + \langle \overline{\mathbf{u}' \bar{\mathbf{u}}} \rangle + \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle - \langle \overline{\mathbf{u}' \mathbf{u}'} \rangle^i] \} &= -\nabla (\phi \langle p' \rangle^i) + \mu \nabla^2 (\phi \langle \mathbf{u}' \rangle^i) + \mathbf{R}' \end{aligned} \quad (49)$$

where  $R'$  is the fluctuating part of the total drag due to the porous structure.

Expanding further the divergent operators in Eq. (49) by means of the expression set (17), one ends up with an equation for  $\langle \mathbf{u}' \rangle^i$  as follows,

$$\begin{aligned} & \rho \frac{\partial}{\partial t} (\phi \langle \mathbf{u}' \rangle^i) + \rho \nabla \cdot \{ \phi [ \langle \bar{\mathbf{u}} \rangle^i \langle \mathbf{u}' \rangle^i + \langle \mathbf{u}' \rangle^i \langle \bar{\mathbf{u}} \rangle^i \\ & + \langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i + \langle \bar{\mathbf{u}}^i \mathbf{u}' \rangle^i + \langle \mathbf{u}'^i \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}'^i \mathbf{u}' \rangle^i \\ & - \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} - \langle \bar{\mathbf{u}}^i \mathbf{u}' \rangle^i \} \\ & = -\nabla (\phi \langle \rho' \rangle^i) + \mu \nabla^2 (\phi \langle \mathbf{u}' \rangle^i) + \mathbf{R}' \end{aligned} \quad (50)$$

## 6 Turbulent kinetic energy

As mentioned, the determination of the flow macroscopic turbulent kinetic energy follows two different paths in the literature. In the models of Refs. [12-15], their turbulence kinetic energy was based on  $k_m = \overline{\langle \mathbf{u}' \rangle^i \cdot \langle \mathbf{u}' \rangle^i} / 2$ . They started with a simplified form of Eq. (50) neglecting the 5th, 6th, 7th, and 9th terms (dispersion). Then they took the scalar product of it with  $\langle \mathbf{u}' \rangle^i$  and applied the time-average operator. On the other hand, if one starts with Eq. (48) and proceed with time-averaging first, one ends up, after volume averaging, with  $\langle k \rangle^i = \overline{\langle \mathbf{u}' \cdot \mathbf{u}' \rangle^i} / 2$ . This was the path followed by Refs. [7, 8, 10]. The objective of this section is to derive both transport equations for  $k_m$  and  $\langle k \rangle^i$  in order to compare similar terms.

**Equation for  $k_m = \overline{\langle \mathbf{u}' \rangle^i \cdot \langle \mathbf{u}' \rangle^i} / 2$ .** From the instantaneous microscopic continuity equation for a constant property fluid one obtains,

$$\nabla \cdot (\phi \langle \mathbf{u}' \rangle^i) = 0 \rightarrow \nabla \cdot [\phi (\langle \bar{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i)] = 0 \quad (51)$$

with time average,

$$\nabla \cdot (\phi \langle \bar{\mathbf{u}} \rangle^i) = 0 \quad (52)$$

From Eq. (51) and (52) one obtains,

$$\nabla \cdot (\phi \langle \mathbf{u}' \rangle^i) = 0 \quad (53)$$

Taking the scalar product of Eq. (49) with  $\langle \mathbf{u}' \rangle^i$ , making use of Eqs. (51)-(53) and time averaging it, an equation for  $k_m$  will have the final form,

$$\begin{aligned} & \rho \frac{\partial (\phi k_m)}{\partial t} + \rho \nabla \cdot [\phi \langle \bar{\mathbf{u}} \rangle^i k_m] \\ & = -\rho \nabla \cdot \left\{ \overline{\phi \langle \mathbf{u}' \rangle^i \left[ \frac{\langle \rho' \rangle^i}{\rho} + \frac{\langle \mathbf{u}' \rangle^i \cdot \langle \mathbf{u}' \rangle^i}{2} \right]} \right\} \\ & + \mu \nabla^2 (\phi k_m) - \rho \phi \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} : \nabla \langle \bar{\mathbf{u}} \rangle^i - \rho \phi \varepsilon_m - D_m \end{aligned} \quad (54)$$

where  $D_m$  represents the dispersion of  $k_m$ . It is interesting to note that this term can be both negative and positive.

The first term on the right of Eq. (54) represents the turbulent diffusion of  $k_m$  and is normally modeled via a diffusion-

like expression resulting for the transport equation for  $k_m$  [14, 15],

$$\begin{aligned} & \rho \frac{\partial (\phi k_m)}{\partial t} + \rho \nabla \cdot [\phi \langle \bar{\mathbf{u}} \rangle^i k_m] = \nabla \cdot \left[ \left( \mu + \frac{\mu_{t_m}}{\sigma_{k_m}} \right) \nabla (\phi k_m) \right] \\ & + P_m - \rho \phi \varepsilon_m - D_m \end{aligned} \quad (55)$$

where

$$P_m = -\rho \phi \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} : \nabla \langle \bar{\mathbf{u}} \rangle^i \quad (56)$$

is the production rate of  $k_m$  due to the gradient of the macroscopic time-mean velocity  $\langle \bar{\mathbf{u}} \rangle^i$ .

References [13-15] made use of the above equation for  $k_m$  considering for  $\mathbf{R}'$  the Darcy-Forchheimer extended model with macroscopic time-fluctuation velocities  $\langle \mathbf{u}' \rangle^i$ . They have also neglected all dispersion terms that were grouped into  $D_m$ . Note also that the order of application of both volume- and time-average operators in this case cannot be changed. The quantity  $k_m$  is defined by applying first the volume operator to the fluctuating velocity field.

**Equation for  $\langle k \rangle^i = \overline{\langle \mathbf{u}' \cdot \mathbf{u}' \rangle^i} / 2$ .** The other procedure for composing the flow turbulent kinetic energy is to take the scalar product of Eq. (48) by the microscopic fluctuating velocity  $\mathbf{u}'$ . Then apply both time and volume-operators for obtaining an equation for  $\langle k \rangle^i = \overline{\langle \mathbf{u}' \cdot \mathbf{u}' \rangle^i} / 2$ . It is worth noting that in this case the order of application of both operations is immaterial since no additional mathematical operation (the scalar product) is conducted between the averaging processes. Therefore, this is the same as applying the volume operator to an equation for the microscopic  $k$ .

The volumetric average of a transport equation for  $k$  has been carried out in detail by Ref [17] and for only that final resulting equation is presented, namely,

$$\begin{aligned} & \rho \frac{\partial}{\partial t} (\phi \langle k \rangle^i) + \nabla \cdot (\bar{\mathbf{u}}_D \langle k \rangle^i) \\ & = \nabla \cdot \left[ \left( \mu + \frac{\mu_{t_k}}{\sigma_k} \right) \nabla (\phi \langle k \rangle^i) \right] + P_i + G_i - \rho \phi \langle \varepsilon \rangle^i \end{aligned} \quad (57)$$

where

$$P_i = -\rho \overline{\langle \mathbf{u}' \mathbf{u}' \rangle^i} : \nabla \bar{\mathbf{u}}_D, \quad G_i = c_k \rho \phi \frac{\langle k \rangle^i |\bar{\mathbf{u}}_D|}{\sqrt{K}} \quad (58)$$

are the production rate of  $\langle k \rangle^i$  due to mean gradients of the seepage velocity  $\bar{\mathbf{u}}_D$  and the generation rate of intrinsic  $k$  due to the presence of the porous matrix, respectively. Also, in Eq. (58)  $K$  is the medium permeability and  $c_k$  is a constant. As mentioned, Eq. (57) has been proposed by Ref [17]. Nevertheless, for the sake of completeness, a few steps of such derivation are here reproduced. Application of the volume average theorem to the transport equation for the turbulence kinetic energy  $k$  gives:

$$\begin{aligned} & \rho \left[ \frac{\partial}{\partial t} (\phi \langle k \rangle^i) + \nabla \cdot (\phi \langle \bar{\mathbf{u}} \rangle^i k) \right] \\ & = -\rho \nabla \cdot \left\{ \phi \left\langle \mathbf{u}' \left( \frac{\rho'}{\rho} + k \right) \right\rangle^i \right\} \end{aligned}$$

$$+\mu \nabla^2(\phi \langle k \rangle^i) - \rho \phi \langle \overline{\mathbf{u}'\mathbf{u}'} : \nabla \bar{\mathbf{u}} \rangle^i - \rho \phi \langle \varepsilon \rangle^i. \quad (59)$$

where the divergence of the right hand side can be expanded as,

$$\nabla \cdot (\phi \langle \bar{\mathbf{u}} k \rangle^i) = \nabla \cdot [\phi (\langle \bar{\mathbf{u}} \rangle^i \langle k \rangle^i + \langle \bar{\mathbf{u}}' k \rangle^i)] \quad (60)$$

where the first term is the convection of  $\langle k \rangle^i$  due to the macroscopic velocity whereas the second is the convective transport due to spatial deviations of both  $k$  and  $\mathbf{u}$ . Likewise, the production term on the right of Eq. (59) can be expanded as,

$$-\rho \phi \langle \overline{\mathbf{u}'\mathbf{u}'} : \nabla \bar{\mathbf{u}} \rangle^i = -\rho \phi [\langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \langle \nabla \bar{\mathbf{u}} \rangle^i + \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \langle \nabla \bar{\mathbf{u}} \rangle^i] \quad (61)$$

Similarly, the first term on the right of Eq. (61) is the production of  $\langle k \rangle^i$  due to the mean macroscopic flow and the second is the  $\langle k \rangle^i$  production associated with spatial deviations of flow quantities  $k$  and  $\mathbf{u}$ .

The extra terms appearing in Eq. (60) and (61), respectively, represent extra transport/production of  $\langle k \rangle^i$  due to the presence of solid material inside the integration volume. They should be null for the limiting case of clear fluid flow, or say, when  $\phi \rightarrow 1 \rightarrow K \rightarrow \infty$ . Also, they should be proportional to the macroscopic velocity and to  $\langle k \rangle^i$ .

In Ref [17], a proposal for those two extra transport/production rates of  $\langle k \rangle^i$  was made as follows:

$$-\nabla \cdot (\phi \langle \bar{\mathbf{u}}' k \rangle^i) - \rho \phi \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \langle \nabla \bar{\mathbf{u}} \rangle^i = G_i = c_k \rho \phi \frac{\langle k \rangle^i |\bar{\mathbf{u}}_D|}{\sqrt{K}} \quad (62)$$

where the constant  $c_k$  was numerically determined by fine flow computations considering the medium to be formed by circular rods [18], as well as longitudinal [19] and transversal rods [20]. In spite of the variation in the medium morphology and the use of a wide range of porosity and Reynolds number a value of 0.28 was found to be suitable for most calculations.

**Comparison of macroscopic transport equations.** A comparison between terms in the transport equation for  $k_m$  and  $\langle k \rangle^i$  can now be conducted. Reference [16] has already shown the connection between these two quantities as being,

$$\langle k \rangle^i = \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i / 2 = \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i / 2 + \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i / 2 = k_m + \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i / 2 \quad (63)$$

Expanding the correlation forming the production term  $P_i$  by means of Eq. (7), a connection between the two generation rates can also be written as follows:

$$\begin{aligned} P_i &= -\rho \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \nabla \bar{\mathbf{u}}_D = -\rho \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \nabla \bar{\mathbf{u}}_D + \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \nabla \bar{\mathbf{u}}_D \\ &= P_m - \rho \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \nabla \bar{\mathbf{u}}_D \end{aligned} \quad (64)$$

We note that all the production rate of  $k_m$ , due to the mean flow, constitutes only part of the general production rate responsible for maintaining the overall level of  $\langle k \rangle^i$ .

## 7 Concluding remarks

In this paper we have described a new methodology for the analysis of turbulent flow in permeable media. A novel concept, called the double-decomposition idea, was detailed

showing how a variable can be decomposed in both time and volume in order to simultaneously account for fluctuations (in time) and deviations (in space) around mean values. Transport equations for the mean and turbulence flow have been presented.

## Acknowledgments

The author would like to express his thanks to CNPq and FAPESP, Brazil, for their financial support during the preparation of this work. Thanks are also due to the authors' former and current graduate students who conducted their graduate programs under the topic here discussed.

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Received in final form: August 11, 2005 [T 910]