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Analysis of turbulent flows in fixed and moving permeable media

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Abstract

The ability to realistically model flows through heterogeneous domains, which contain both solid and fluid phases, can benefit the analysis and simulation of complex real-world systems. Environmental impact studies, as well as engineering equipment design, can both take advantage of reliable modelling of turbulent flow in permeable media. Turbulence models proposed for such flows depend on the order of application of volume- and time-average operators. Two methodologies, following the two orders of integration, lead to distinct governing equations for the statistical quantities. This paper reviews recently published methodologies to mathematically characterize turbulent transport in permeable media.

A new concept, called double-decomposition, is here discussed and instantaneous local transport equations are reviewed for clear flow before the time and volume averaging procedures are applied to them. Equations for turbulent transport follow, including their detailed derivation and a proposed model for suitable numerical simulations. The case of a moving porous bed is also discussed and transport equations for the mean and turbulent flow fields are presented.

Key words: turbulent flow, porous media, moving bed.

1. INTRODUCTION

A number of natural and engineering systems can be characterized by a permeable structure through which a working fluid permeates. For treating and simulating such complex systems, it is often convenient and economical to resort to tools that can gather the most information using the minimum computational resources. To achieve this goal, it is generally recognized that the use of an appropriate macroscopic view of such complex systems which reduce computational effort while maintaining a substantial level of information, can bring an adequate compromise between completeness and economics of a realistic analysis.

It is generally accepted in the open literature that mathematical modelling of macroscopic transport for incompressible flows in porous media can be based on the volume-average methodology (Whitaker 1999) for either heat (Hsu and Cheng 1990) or mass transfer (Bear 1972, Bear and Bachmat 1967, Whitaker 1966, 1967). Additional important volume- and double-averaging contributions can be found in the specialized literature (Finnigan 1985, Gray 1975, Raupach and Shaw 1982, Wilson and Shaw 1977, Nikora *et al*. 2007). If the fluid phase properties fluctuate in time, in addition to presenting spatial deviations around a volume-mean value, there are two possible methodologies to follow in order to obtain macroscopic equations: (a) application of time-average operator followed by volume-averaging (Masuoka and Takatsu 1996, Takatsu and Masuoka 1998, Kuwahara *et al*. 1996, Kuwahara and Nakayama 1998, Nakayama and Kuwahara 1999), or (b) use of volume-averaging before time-averaging is applied (Lee and Howell 1987, Wang and Takle 1995, Antohe and Lage 1997, Getachewa *et al*. 2000).

In fact, these two sets of macroscopic transport equations are equivalent when examined under the recently established ***double decomposition*** concept (Pedras and de Lemos 2000, 2001, 2003). Recent reviews on the topic of turbulence in permeable media can be found in Lage (1998) and Lage *et al*. (2002). Advances on the general area of porous media are detailed in recently published books (Nield and Bejan 1999, Ingham and Pop 2002, Vafai 2000), in addition to an entire volume dedicated to the topic of turbulent flow in permeable media (de Lemos 2006).

The ***double-decomposition*** idea was initially proposed for the flow variables and has been extended to non-buoyant heat transfer (Rocamora and de Lemos 2000), buoyant flows (de Lemos and Braga 2003, Braga and de Lemos 2004, 2005, 2006a, b), mass transfer (de Lemos and Mesquita 2003), non-equilibrium heat transfer (Saito and de Lemos 2005, 2006), double-diffusive transport (de Lemos and Tofaneli 2004), and hybrid media (clear/porous domains) (Assato *et al*. 2005, Santos and de Lemos 2006). The problem of treating macroscopic interfaces bounding finite porous media, considering a diffusion-jump condition for the mean (Silva and de Lemos 2003a, b) and turbulent flow fields (de Lemos 2005, de Lemos and Silva 2006), has also been investigated under the concept first proposed by Pedras and de Lemos (2001). A general classification of all proposed models for turbulent flow and heat transfer in porous media has been also published by de Lemos and Pedras (2001).

Although the publications refereed above have systematically extended the double-decomposition theory to tackle a wide range of problems, herein a concise introduction to this concept is offered to the reader, extending the previously published work to the case of a moving porous bed.

2. AVERAGING OPERATORS AND TRANSPORT EQUATIONS

The steady-state local or microscopic instantaneous transport equations for an incompressible fluid with constant properties are given by:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho \nabla \cdot (\mathbf{uu}) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}, \tag{2}$$

where $\mathbf{u}$ is the velocity vector, ρ is the density, p is the pressure, μ is the fluid viscosity and $\mathbf{g}$ is the gravity acceleration vector.

As mentioned, there are, in principle, two ways that one can follow in order to treat the turbulent flow in porous media. The first method applies a time-average operator to the governing eqs. (1) and (2) before the volume-average procedure is applied. In the second approach, the order of application of the two average operators is reversed. Both techniques aim at derivation of suitable macroscopic transport equations.

Volume averaging in a porous medium, described in detail in Slattery (1967), Whitaker (1969), Gray and Lee (1977), makes use of the concept of a Representative Elementary Volume (REV) over which local equations are integrated. In a similar fashion, statistical analysis of turbulent flow leads to time mean properties. Transport equations for statistical values are considered in lieu of instantaneous information on the flow.

Time and volume averaging procedures

Traditional analyses of turbulence are based on statistical quantities, which are obtained by applying time-averaging to the flow governing equations. As such, the time average of a general quantity φ is defined as follows (not to confuse with the porosity ϕ to be defined later):

$$\overline{\varphi} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \varphi \, \mathrm{d}t, \tag{3}$$

where the time interval Δt is small compared to the fluctuations of the average value, $\overline{\varphi}$, but large enough to capture turbulent fluctuations of φ. Time decomposition can then be written as follows:

$$\varphi = \overline{\varphi} + \varphi', \tag{4}$$

with $\overline{\varphi'} = 0$, where φ' is the time fluctuation of φ around its average value $\overline{\varphi}$.

The volume average of a general property φ taken over a REV in a porous medium can be written as in (Slattery 1967),

$$\langle\varphi\rangle^v = \frac{1}{\Delta V}\int_{\Delta V} \varphi \,\mathrm{d}V. \tag{5}$$

The value $\langle\varphi\rangle^v$ is defined for any point $\mathbf{x}$ surrounded by a REV of size ΔV. This average is related to the intrinsic average for the fluid phase as follows:

$$\langle\varphi\rangle^v = \phi\langle\varphi\rangle^i, \tag{6}$$

where $\phi = \Delta V_f / \Delta V$ is the local medium porosity and ΔV_f is the volume occupied by the fluid in a REV. Furthermore, we can write,

$$\varphi = \langle\varphi\rangle^i + {}^i\varphi, \tag{7}$$

with $\langle {}^i\varphi\rangle^i = 0$. In eq. (7), ${}^i\varphi$ is the spatial deviation of φ with respect to the intrinsic average $\langle\varphi\rangle^i$.

For deriving the flow governing equations, it is necessary to know the relationship between the volumetric average of derivatives and the derivatives of the volumetric average. These relationships are presented in a number of works (e.g., Whitaker 1969, 1999), being known as the Theorem of Local Volumetric Average. They are written as follows:

$$\langle\nabla\varphi\rangle^v = \nabla\left(\phi\langle\varphi\rangle^i\right) + \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\varphi \,\mathrm{d}S, \tag{8}$$

$$\langle\nabla\cdot\boldsymbol{\varphi}\rangle^v = \nabla\cdot\left(\phi\langle\boldsymbol{\varphi}\rangle^i\right) + \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\cdot\boldsymbol{\varphi} \,\mathrm{d}S, \tag{9}$$

$$\left\langle\frac{\partial\varphi}{\partial t}\right\rangle^v = \frac{\partial}{\partial t}\left(\phi\langle\phi\rangle^i\right) - \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\cdot(\mathbf{u}_i\varphi) \,\mathrm{d}S, \tag{10}$$

where A_i, $\mathbf{u}_i$ and $\mathbf{n}$ are the interfacial area, the velocity of the interface and the outward unit vector normal to A_i, respectively.

The area A_i should not be confused with the surface area surrounding volume ΔV. To the interested reader, mathematical details and proof of the Theorem of Local Volumetric Average can be found in: Whitaker (1969, 1999), Slattery (1967), Gray and Lee (1977). For single-phase flow, phase f is the fluid itself and $\mathbf{u}_i = 0$ if the porous substrate is assumed to be fixed. In developing eqs. (8)-(10), the only restriction applied is the independence of ΔV in relation to time and space. If the medium is further assumed to be rigid, then ΔV_f is dependent only on space and not time-dependent (Gray and Lee 1977).

Time averaged transport equations

In order to apply the time-average operator to eqs. (1), (2) and (8), we assume:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \qquad p = \overline{p} + p'. \tag{11}$$

Substituting expression (11) into eqs. (1), (2) and (8), we obtain, after considering constant flow properties,

$$\nabla \cdot \overline{\mathbf{u}} = 0, \tag{12}$$

$$\rho \nabla \cdot (\overline{\mathbf{u}}\overline{\mathbf{u}}) = -(\nabla \overline{p})^* + \mu \nabla^2 \overline{\mathbf{u}} + \nabla \cdot (-\rho \overline{\mathbf{u}'\mathbf{u}'}). \tag{13}$$

For a clear fluid, the use of the eddy-diffusivity concept for expressing the stress-rate of strain relationship for the Reynolds stress appearing in eq. (13) gives

$$-\rho \overline{\mathbf{u}'\mathbf{u}'} = \mu_t 2\overline{\mathbf{D}} - \frac{2}{3}\rho k \mathbf{I}, \tag{14}$$

where $\overline{\mathbf{D}}$ is the mean deformation tensor, k is the turbulent kinetic energy per unit mass, μ_t is the turbulent viscosity and $\mathbf{I}$ is the unity tensor.

The transport equation for the turbulent kinetic energy is obtained by multiplying first the difference between the instantaneous and the time-averaged momentum equations by $\mathbf{u}'$. Thus, applying further the time-average operator to the resulting product, we obtain

$$\rho \nabla \cdot (\overline{\mathbf{u}}k) = -\rho \nabla \cdot \overline{\left[\mathbf{u}'\left(\frac{p'}{\rho} + q\right)\right]} + \mu \nabla^2 k + P_k - \rho\varepsilon, \tag{15}$$

where $P_k = -\rho \overline{\mathbf{u}'\mathbf{u}'} : \nabla \overline{\mathbf{u}}$ is the generation rate of k due to gradients of the mean velocity and $q = \mathbf{u}' \cdot \mathbf{u}'/2$.

3. DOUBLE DECOMPOSITION

The double decomposition idea, herein used for obtaining macroscopic equations, has been detailed in (Pedras and de Lemos 2000, 2001, 2003). Here, a general overview is presented. Further, the resulting equations using this concept for the flow (Pedras and de Lemos 2001) and non-buoyant thermal fields (Rocamora and de Lemos 2000) are already available in the literature and because of this they are not reviewed here in greater detail. As already mentioned, extensions of the double-decomposition methodology to buoyant flows (de Lemos and Braga 2003, Braga and de Lemos 2004), to mass transport (de Lemos and Mesquita 2003), and to double-diffusive convection (de Lemos and Tofaneli 2004), have also been presented in the open literature.

Basically, for porous media analysis, a macroscopic form of the governing equations is obtained by taking the volumetric average of the entire equation set. In that development, the porous medium is considered to be rigid and saturated by an incompressible fluid.

Basic relationships

From the work presented by Pedras and de Lemos (2000), and Rocamora Jr. and de Lemos (2000), one can write for any flow property φ, combining decompositions (7) and (4),

$$\langle\varphi\rangle^i = \langle\overline{\varphi}\rangle^i + \langle\varphi'\rangle^i , \qquad \overline{\varphi} = \langle\overline{\varphi}\rangle^i + {}^i\overline{\varphi} ,$$
$$ {}^i\varphi = {}^i\overline{\varphi} + {}^i\varphi' , \qquad \varphi' = \langle\varphi'\rangle^i + {}^i\varphi' \tag{16}$$

or further

$$\varphi' = \langle\varphi\rangle^{i'} + {}^i\varphi' = \langle\varphi'\rangle^i + {}^i\varphi', \tag{17}$$

where ${}^i\varphi'$ can be understood as either the time fluctuation of the spatial deviation or the spatial deviation of the time fluctuation. After some manipulation, we can prove that (Pedras and de Lemos 2001),

$$\overline{\langle\varphi\rangle^v} = \langle\overline{\varphi}\rangle^v \quad \text{or} \quad \overline{\langle\varphi\rangle^i} = \langle\overline{\varphi}\rangle^i , \tag{18}$$

i.e., the time and volume averages commute. Also,

$$ {}^i\overline{\varphi} = \overline{{}^i\varphi}; \qquad \langle\varphi'\rangle^i = \langle\varphi\rangle^{i'} \tag{19}$$

or say,

$$\langle\varphi\rangle^i = \frac{1}{\Delta V_f}\int_{\Delta V_f} \varphi \, \mathrm{d}V = \frac{1}{\Delta V_f}\int_{\Delta V_f} (\overline{\varphi} + \varphi')\, \mathrm{d}V = \langle\overline{\varphi}\rangle^i + \langle\varphi'\rangle^i , \tag{20}$$

$$ {}^i\varphi = {}^i(\overline{\varphi}) + {}^i(\varphi') = \overline{({}^i\varphi)} + ({}^i\varphi)' \tag{21}$$

so that,

$$\varphi' = \langle\varphi'\rangle^i + {}^i(\varphi'),$$

where

$$ {}^i(\varphi') = \varphi' - \langle\varphi'\rangle^i \ \text{ and also } \ ({}^i\varphi)' = {}^i\varphi - \overline{{}^i\varphi} . \tag{22}$$

Finally, we can have a full variable decomposition as follows:

$$\varphi = \langle\overline{\varphi}\rangle^i + \langle\varphi'\rangle^i + {}^i\overline{\varphi} + {}^i(\varphi') = \overline{\langle\varphi\rangle^i} + \langle\varphi\rangle^{i'} + \overline{{}^i\varphi} + ({}^i\varphi)' \tag{23}$$

or further,

$$\varphi = \underbrace{\langle\overline{\varphi}\rangle^i + \langle\varphi'\rangle^i}_{\langle\varphi\rangle^i} + \overbrace{{}^i\overline{\varphi} + {}^i(\varphi')}^{{}^i\varphi} = \underbrace{\overline{\langle\varphi\rangle^i} + \overline{{}^i\varphi}}_{\overline{\varphi}} + \overbrace{\langle\varphi\rangle^{i'} + ({}^i\varphi)'}^{\varphi'} . \tag{24}$$

Equation (23) comprises the double-decomposition concept. The significance of the four terms in expression (24) can be reviewed as:

(a) $\langle\overline{\varphi}\rangle^i$ is the intrinsic average of the time mean value of φ, i.e., we compute first the time averaged values of all points composing the REV, and then we find their volumetric mean to obtain $\langle\overline{\varphi}\rangle^i$. Instead, we could also consider a certain point **x** surrounded by the REV, according to eqs. (5) and (6), and take the volumetric average at different time steps. Thus, we calculate the average over such different values of $\langle \varphi \rangle^i$ in time. We get then $\overline{\langle \varphi \rangle^i}$ and, according to expression (18), $\langle \overline{\varphi} \rangle^i = \overline{\langle \varphi \rangle^i}$, i.e., the volumetric and time average commute.

(b) If we now take the volume average of all fluctuating components of φ, which compose the REV, we end up with $\langle \varphi' \rangle^i$. Instead, with the volumetric average around point **x** taken at different time steps we can determine the difference between the instantaneous and a time averaged value. This will be $\langle \varphi \rangle^{i\prime}$ that, according to expression (19), equals $\langle \varphi' \rangle^i$. Further, on performing first a time-averaging operation over all points that contribute with their local values to the REV, we get a distribution of $\overline{\varphi}$ within this volume. If we now calculate the intrinsic average of this distribution of $\overline{\varphi}$, we get $\langle \overline{\varphi} \rangle^i$. The difference or deviation between these two values is ${}^i\overline{\varphi}$. Now, using the same space decomposition approach, we can find the deviation ${}^i\varphi$ for any instant of time t. This value also fluctuates with time, and as such a time mean can be calculated as $\overline{{}^i\varphi}$. Again the use of expression (19) gives ${}^i\overline{\varphi} = \overline{{}^i\varphi}$. Finally, it is interesting to note the meaning of the last term on each side of eq. (24). The first term, ${}^i(\varphi')$, is the time fluctuation of the spatial deviation whereas $({}^i\varphi)'$ means the spatial deviation of the time varying term. If, however, one makes use of relationships (18) and (19) to simplify expression (24), we finally conclude,

$$ {}^i(\varphi') = ({}^i\varphi)' \tag{25} $$

and, for simplicity of notation, we can drop the parentheses and write both superscripts at the same level in the format: ${}^i\varphi'$. Also, $\langle {}^i\varphi' \rangle^i = \overline{{}^i\varphi'} = 0$.

The basic advantage of the double decomposition concept is to serve as a mathematical framework for analysis of flows where within the fluid phase there is enough room for turbulence to be established. As such, the double-decomposition methodology would be useful in situations where a solid phase is present in the domain under analysis so that a macroscopic view is appropriate. At the same time, properties in the fluid phase are subjected to the turbulent regime, and a statistical approach is appropriate. Examples of possible applications of such a methodology can be found in engineering systems such as heat exchangers, porous combustors, nuclear reactor cores, etc. Natural systems include atmospheric boundary layer over forests and crops.

4. EQUATIONS FOR MASS AND MOMENTUM

The development to follow assumes single-phase flow in a saturated, rigid porous medium (ΔV_f independent of time) for which, in accordance with expression (18), time average operation on the variable φ commutes with the space average. Application of the double-decomposition idea in eq. (24) to the inertia term in the momentum equation leads to four different terms. Not all of these terms are considered in the same analysis in the literature.

Continuity

The microscopic continuity equation for an incompressible fluid flowing in a clear (non-porous) domain was given by eq. (1). Using the double-decomposition idea embodied in expression (24) results,

$$\nabla \cdot \mathbf{u} = \nabla \cdot \left(\langle \overline{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i + {}^i\overline{\mathbf{u}} + {}^i\mathbf{u}' \right) = 0. \tag{26}$$

On applying both volume- and time-average operators in either order gives

$$\nabla \cdot \left(\phi \langle \overline{\mathbf{u}} \rangle^i \right) = 0. \tag{27}$$

As such, for the continuity equation the averaging order is immaterial.

Momentum – one average operator

The transient form of the microscopic momentum eq. (2) for a fluid with constant properties is given by the Navier-Stokes equation as follows:

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) \right] = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}. \tag{28}$$

Its time-average, using $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, gives

$$\rho \left[\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot \left(\overline{\mathbf{u}}\,\overline{\mathbf{u}} \right) \right] = -\nabla \overline{p} + \mu \nabla^2 \overline{\mathbf{u}} + \nabla \cdot \left(-\rho \overline{\mathbf{u}'\mathbf{u}'} \right) + \rho \mathbf{g}, \tag{29}$$

where the stresses, $-\rho \overline{\mathbf{u}'\mathbf{u}'}$, are the well-known Reynolds stresses. On the other hand, the volumetric average of eq. (28) using the ***Theorem of Local Volumetric Average***, eqs. (8)-(10), results in the following:

$$\rho \left[\frac{\partial}{\partial t} \left(\phi \langle \mathbf{u} \rangle^i \right) + \nabla \cdot \left[\phi \langle \mathbf{u}\mathbf{u} \rangle^i \right] \right] = -\nabla \left(\phi \langle p \rangle^i \right) + \mu \nabla^2 \left(\phi \langle \mathbf{u} \rangle^i \right) + \phi \rho \mathbf{g} + \mathbf{R}, \tag{30}$$

where

$$\mathbf{R} = \frac{\mu}{\Delta V} \int_{A_i} \mathbf{n} \cdot (\nabla \mathbf{u}) \, \mathrm{d}S - \frac{1}{\Delta V} \int_{A_i} \mathbf{n} \, p \, \mathrm{d}S \tag{31}$$

represents the total drag force per unit volume due to the presence of the porous matrix, being composed by both viscous drag and form (pressure) drags. Further, using spatial decomposition to write $\mathbf{u} = \langle \mathbf{u} \rangle^i + {}^i\mathbf{u}$ in the inertia term we obtain the following:

$$\rho\left[\frac{\partial}{\partial t}\left(\phi\langle\mathbf{u}\rangle^i\right)+\nabla\cdot\left[\phi\langle\mathbf{u}\rangle^i\langle\mathbf{u}\rangle^i\right]\right]$$
$$=-\nabla\left(\phi\langle p\rangle^i\right)+\mu\nabla^2\left(\phi\langle\mathbf{u}\rangle^i\right)-\nabla\cdot\left[\phi\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i\right]+\phi\rho\mathbf{g}+\mathbf{R}. \tag{32}$$

Hsu and Cheng (1990) pointed out that the third term on the right-hand side represents the hydrodynamic dispersion due to spatial deviations. Note that eq. (32) models typical porous media flow for $Re_p < 150\text{-}200$. When extending the analysis to turbulent flow, time varying quantities have to be considered.

Momentum equation – two average operators

The set of eqs. (29) and (32) is used when treating turbulent flow in a clear fluid, or low Re_p porous media flow, respectively. In each of those equations, only one averaging operator was applied, either time or volume, respectively. In this work, an investigation on the use of both operators is now conducted with the objective of modelling turbulent flow in porous media.

The volume average of eq. (29) gives for the time mean flow in a porous medium,

$$\rho\left[\frac{\partial}{\partial t}\left(\phi\langle\overline{\mathbf{u}}\rangle^i\right)+\nabla\cdot\left(\phi\langle\overline{\mathbf{u}}\,\overline{\mathbf{u}}\rangle^i\right)\right]$$
$$=-\left(\phi\langle\overline{p}\rangle^i\right)+\mu\nabla^2\left(\phi\langle\overline{\mathbf{u}}\rangle^i\right)+\nabla\cdot\left(-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i\right)+\phi\rho\mathbf{g}+\overline{\mathbf{R}}, \tag{33}$$

where

$$\overline{\mathbf{R}}=\frac{\mu}{\Delta V}\int_{A_i}\mathbf{n}\cdot(\nabla\overline{\mathbf{u}})\,\mathrm{d}S-\frac{1}{\Delta V}\int_{A_i}\mathbf{n}\,\overline{p}\,\mathrm{d}S \tag{34}$$

is the time-averaged total drag force per unit volume due to solid particles, composed by both viscous and form (pressure) drags.

Likewise, applying now the time-average operation to eq. (30), we obtain

$$\rho\left[\frac{\partial}{\partial t}\overline{(\phi\langle\overline{\mathbf{u}}+\mathbf{u}'\rangle^i)}+\nabla\cdot\overline{(\phi\langle(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')\rangle^i)}\right]$$
$$=-\nabla\overline{(\phi\langle\overline{p}+p'\rangle^i)}+\mu\nabla^2\overline{(\phi\langle\overline{\mathbf{u}}+\mathbf{u}'\rangle^i)}+\phi\rho\mathbf{g}+\overline{\mathbf{R}}. \tag{35}$$

Dropping the terms containing only one fluctuating quantity results in

$$\rho\left[\frac{\partial}{\partial t}\left(\phi\langle\overline{\mathbf{u}}\rangle^i\right)+\nabla\cdot\left(\phi\langle\overline{\mathbf{u}}\,\overline{\mathbf{u}}\rangle^i\right)\right]$$
$$=-\nabla\left(\phi\langle\overline{p}\rangle^i\right)+\mu\nabla^2\left(\phi\langle\overline{\mathbf{u}}\rangle^i\right)+\nabla\cdot\left(-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i\right)+\phi\rho\mathbf{g}+\overline{\mathbf{R}}, \tag{36}$$

where

$$\overline{\mathbf{R}} = \frac{\mu}{\Delta V}\int_{A_i} \mathbf{n}\cdot\left[\overline{\nabla(\overline{\mathbf{u}}+\mathbf{u}')}\right]\mathrm{d}S - \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\,\overline{(\overline{p}+p')}\,\mathrm{d}S$$

$$= \frac{\mu}{\Delta V}\int_{A_i} \mathbf{n}\cdot(\nabla\overline{\mathbf{u}})\,\mathrm{d}S - \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\,\overline{p}\,\mathrm{d}S. \tag{37}$$

Comparing eqs. (33) and (36), we can see that also for the momentum equation the order of the application of both averaging operators is immaterial.

It is interesting to emphasize that both views in the literature use the same final form for the momentum equation. The term $\overline{\mathbf{R}}$ is modelled by the Darcy-Forchheimer (Dupuit) expression after either order of application of the average operators. Since both orders of integration lead to the same equation, namely expression (34) or (37), there would be no reason for modelling them in a different form. Had the outcome of both integration processes been distinct, the use of a different model for each case would have been consistent. In fact, it has been pointed out by Pedras and de Lemos (2000) that the major difference between those two paths lies in the definition of a suitable turbulent kinetic energy for the flow. Accordingly, the source of controversies comes from the inertia term, as seen below.

Inertia term – double decomposition

Applying to the inertia term of eq. (28) the double decomposition idea seen before for velocity (eq. (24)), will lead to different sets of terms. In the literature, not all of them are used in the same analysis.

Starting with time decomposition and applying both average operators (see, eq. 33) gives

$$\nabla\cdot\overline{(\phi\langle\mathbf{uu}\rangle^i)} = \nabla\cdot\overline{(\phi\langle(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')\rangle^i)} = \nabla\cdot[\phi(\langle\overline{\mathbf{u}}\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i)] \tag{38}$$

using the spatial decomposition to write $\overline{\mathbf{u}} = \langle\overline{\mathbf{u}}\rangle^i + {}^i\overline{\mathbf{u}}$ we obtain

$$\nabla\cdot[\phi(\langle\overline{\mathbf{u}}\,\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i)] = \nabla\cdot\left\{\phi[\langle(\langle\overline{\mathbf{u}}\rangle^i + {}^i\overline{\mathbf{u}})(\langle\overline{\mathbf{u}}\rangle^i + {}^i\overline{\mathbf{u}})\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i]\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i]\right\}. \tag{39}$$

Now, applying eq. (17) to write $\mathbf{u}' = \langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}'$, and substituting into expression (39) gives

$$\nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i]\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle(\langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}')(\langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}')\rangle^i}]\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle(\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i + \langle\mathbf{u}'\rangle^i\,{}^i\mathbf{u}' + {}^i\mathbf{u}'\langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}'\,{}^i\mathbf{u}')\rangle^i}]\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + \overline{\langle\langle\mathbf{u}'\rangle^i\,{}^i\mathbf{u}'\rangle^i} + \overline{\langle{}^i\mathbf{u}'\langle\mathbf{u}'\rangle^i\rangle^i} + \overline{\langle{}^i\mathbf{u}'\,{}^i\mathbf{u}'\rangle^i}]\right\}. \quad (40)$$

The fourth and fifth terms on the right-hand side contain only one space-varying quantity and will vanish under the application of volume integration. Equation (40) will then be reduced to

$$\nabla\cdot\overline{(\phi\langle\mathbf{u}\mathbf{u}\rangle^i)} = \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle{}^i\mathbf{u}'\,{}^i\mathbf{u}'\rangle^i}]\right\}. \quad (41)$$

Using the equivalence (18)-(19), eq. (41) can be further rewritten as follows:

$$\nabla\cdot\overline{(\phi\langle\mathbf{u}\mathbf{u}\rangle^i)} = \nabla\cdot\left\{\phi[\overline{\langle\mathbf{u}\rangle^i}\,\overline{\langle\mathbf{u}\rangle^i} + \overline{\langle\mathbf{u}\rangle^{i\prime}\langle\mathbf{u}\rangle^{i\prime}} + \langle\overline{{}^i\mathbf{u}}\,\overline{{}^i\mathbf{u}}\rangle^i + \langle\overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i]\right\} \quad (42)$$

with an interpretation of the terms in eq. (41) given later.

Another route to follow to reach the same results is to start out with an application of the space decomposition in the inertia term, as usually done in classical mathematical treatment of porous media flow analysis. Then we obtain

$$\nabla\cdot\overline{(\phi\langle\mathbf{u}\mathbf{u}\rangle^i)} = \nabla\cdot\overline{(\phi\langle((\langle\mathbf{u}\rangle^i + {}^i\mathbf{u})(\langle\mathbf{u}\rangle^i + {}^i\mathbf{u}))\rangle^i)} = \nabla\cdot\overline{[\phi(\langle\mathbf{u}\rangle^i\langle\mathbf{u}\rangle^i + \langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i)]} \quad (43)$$

and on time averaging the r.h.s., using eq. (20) to express $\langle\mathbf{u}\rangle^i = \langle\overline{\mathbf{u}}\rangle^i + \langle\mathbf{u}'\rangle^i$, becomes

$$\nabla\cdot\overline{[\phi(\langle\mathbf{u}\rangle^i\langle\mathbf{u}\rangle^i + \langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i)]} = \nabla\cdot\left\{\overline{\phi[(\langle\overline{\mathbf{u}}\rangle^i + \langle\mathbf{u}'\rangle^i)(\langle\overline{\mathbf{u}}\rangle^i + \langle\mathbf{u}'\rangle^i) + \langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i]}\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + \overline{\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i}]\right\}. \quad (44)$$

With the help of eq. (21) one can write ${}^i\mathbf{u} = {}^i\overline{\mathbf{u}} + {}^i\mathbf{u}'$ which, inserted into expression (44), gives

$$\nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}] + \overline{\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i}]\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + \overline{\langle({}^i\overline{\mathbf{u}} + {}^i\mathbf{u}')({}^i\overline{\mathbf{u}} + {}^i\mathbf{u}')\rangle^i}]\right\}$$

$$= \nabla\cdot\left\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + \langle\overline{{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}} + \overline{{}^i\overline{\mathbf{u}}\,{}^i\mathbf{u}'} + \overline{{}^i\mathbf{u}'\,{}^i\overline{\mathbf{u}}} + \overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i]\right\}. \quad (45)$$

Application of the time-average operator to the fourth and fifth terms on the right-hand side of eq. (45), containing only one fluctuating component, vanishes it. In addition, remembering that with expression (19) the equivalences ${}^i\overline{\mathbf{u}} = \overline{{}^i\mathbf{u}}$ and $\langle \mathbf{u}' \rangle^i = \langle \mathbf{u} \rangle^{i\prime}$ are valid, and that with expression (18) we can write $\overline{\langle \mathbf{u} \rangle^i} = \langle \overline{\mathbf{u}} \rangle^i$, we obtain the following alternative form for eq. (45):

$$\nabla \cdot \overline{[\phi(\langle \mathbf{u} \rangle^i \langle \mathbf{u} \rangle^i + \langle {}^i\mathbf{u}\, {}^i\mathbf{u} \rangle^i)]} = \nabla \cdot \left\{ \phi[\underbrace{\langle \overline{\mathbf{u}} \rangle^i \langle \overline{\mathbf{u}} \rangle^i}_{\text{I}} + \underbrace{\overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i}}_{\text{II}} + \underbrace{\langle {}^i\overline{\mathbf{u}}\, {}^i\overline{\mathbf{u}} \rangle^i}_{\text{III}} + \underbrace{\overline{\langle {}^i\mathbf{u}'\, {}^i\mathbf{u}' \rangle^i}}_{\text{IV}}] \right\}, \quad (46)$$

which is the same result as expression (41).

The physical significance of all four terms on the right-hand side of (46) can be discussed as:

- ***Convective*** term of macroscopic mean velocity.
- ***Turbulent (Reynolds) stresses*** divided by the density ρ due to the fluctuating component of the macroscopic velocity.
- ***Dispersion*** associated with spatial fluctuations of microscopic time mean velocity. Note that this term is also present in the laminar flow, or say, when the pore-based Reynolds number, Re_p, is less than 150.
- ***Turbulent dispersion*** in a porous medium due to both time and spatial deviation of the microscopic velocity.

5. TURBULENT KINETIC ENERGY

The starting point for an equation for the flow turbulent kinetic energy is an equation for the microscopic velocity fluctuation $\mathbf{u}'$. Such a relationship can be written, after subtracting the equation for the mean velocity $\overline{\mathbf{u}}$ from the instantaneous momentum equation, in the following form:

$$\rho \left\{ \frac{\partial \mathbf{u}'}{\partial t} + \nabla \cdot [\overline{\mathbf{u}}\,\mathbf{u}' + \mathbf{u}'\,\overline{\mathbf{u}} + \mathbf{u}'\mathbf{u}' - \overline{\mathbf{u}'\mathbf{u}'}] \right\} = -\nabla p' + \mu \nabla^2 \mathbf{u}'. \quad (47)$$

Now, the volumetric average of eq. (47), using the Theorem of Local Volumetric Average, gives

$$\rho \frac{\partial}{\partial t}(\varphi \langle \mathbf{u}' \rangle^i) + \rho \nabla \cdot \left\{ \varphi [\langle \overline{\mathbf{u}}\,\mathbf{u}' \rangle^i + \langle \mathbf{u}'\,\overline{\mathbf{u}} \rangle^i + \langle \mathbf{u}'\mathbf{u}' \rangle^i - \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i] \right\}$$
$$= -\nabla(\varphi \langle p' \rangle^i) + \mu \nabla^2 (\varphi \langle \mathbf{u}' \rangle^i) + \mathbf{R}', \quad (48)$$

where $\mathbf{R}'$ is the fluctuating part of the total drag due to the porous structure.

Expanding further the divergent operators in eq. (48) by means of the expression set (16), one ends up with an equation for $\langle \mathbf{u}' \rangle^i$ as follows:

$$\rho \frac{\partial}{\partial t}(\phi \langle \mathbf{u}' \rangle^i) + \rho \nabla \cdot \left\{ \phi [\langle \overline{\mathbf{u}} \rangle^i \langle \mathbf{u}' \rangle^i + \langle \mathbf{u}' \rangle^i \langle \overline{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i \right.$$
$$\left. + \langle {}^i\overline{\mathbf{u}}\, {}^i\mathbf{u}' \rangle^i + \langle {}^i\mathbf{u}'\, {}^i\overline{\mathbf{u}} \rangle^i + \langle {}^i\mathbf{u}'\, {}^i\mathbf{u}' \rangle^i - \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} - \langle \overline{{}^i\mathbf{u}'\, {}^i\mathbf{u}'} \rangle^i] \right\}$$
$$= -\nabla(\phi \langle p' \rangle^i) + \mu \nabla^2 (\phi \langle \mathbf{u}' \rangle^i) + \mathbf{R}'. \quad (49)$$

As mentioned, the determination of the flow macroscopic turbulent kinetic energy follows two different paths in the literature. In the models of Lee and Howell (1987), Wang and Takle (1995), Antohe and Lage (1997), Getachewa *et al*. (2000), their turbulence kinetic energy was based on $k_m = \overline{\langle \mathbf{u}' \rangle^i \cdot \langle \mathbf{u}' \rangle^i} / 2$. They started with a simplified form of eq. (49) neglecting the 5-, 6-, 7- and 9-th terms (dispersion). Then they took the scalar product of it with $\langle \mathbf{u}' \rangle^i$ and applied the time-average operator.

On the other hand, if one starts with eq. (47) and proceed with time-averaging first, one ends up, after volume averaging, with $\langle k \rangle^i = \langle \overline{\mathbf{u}' \cdot \mathbf{u}'} \rangle^i / 2$. This was the path followed by Masuoka and Takatsu (1996), Takatsu and Masuoka (1998), Kuwahara and Nakayama (1998). The objective of this section is to derive both transport equations, for k_m and $\langle k \rangle^i$, in order to compare similar terms.

Transport equation for k_m

From the instantaneous microscopic continuity equation for a constant-property fluid one obtains

$$\nabla \cdot (\phi \langle \mathbf{u} \rangle^i) = 0 \quad \Rightarrow \quad \nabla \cdot \left[\phi (\langle \overline{\mathbf{u}} \rangle^i + \langle \mathbf{u}' \rangle^i) \right] = 0, \tag{50}$$

with the time average

$$\nabla \cdot (\phi \langle \overline{\mathbf{u}} \rangle^i) = 0. \tag{51}$$

From eqs. (50) and (51) we obtain

$$\nabla \cdot (\phi \langle \mathbf{u}' \rangle^i) = 0. \tag{52}$$

Taking the scalar product of eq. (48) with $\langle \mathbf{u}' \rangle^i$, making use of eqs. (50)-(52) and time averaging it, the equation for k_m will have the final form:

$$\rho \frac{\partial (\phi k_m)}{\partial t} + \rho \nabla \cdot \left[\phi \langle \overline{\mathbf{u}} \rangle^i k_m \right] = -\rho \nabla \cdot \overline{\left\{ \phi \langle \mathbf{u}' \rangle^i \left[\frac{\langle p' \rangle^i}{\rho} + \frac{\langle \mathbf{u}' \rangle^i \cdot \langle \mathbf{u}' \rangle^i}{2} \right] \right\}}$$

$$+ \mu \nabla^2 (\phi k_m) - \rho \phi \overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} : \nabla \langle \overline{\mathbf{u}} \rangle^i - \rho \phi \varepsilon_m - D_m, \tag{53}$$

where D_m represents the dispersion of k_m. It is interesting to note that this term can be both negative and positive.

The first term on the right of eq. (53) represents the turbulent diffusion of k_m and is normally modelled *via* a diffusion-like expression resulting for the transport equation for k_m (Antohe and Lage 1997, Getachewa *et al*. 2000)

$$\rho \frac{\partial (\phi k_m)}{\partial t} + \rho \nabla \cdot [\phi \langle \overline{\mathbf{u}} \rangle^i k_m] = \nabla \cdot \left[\mu + \frac{\mu_{t_m}}{\sigma_{k_m}} \nabla (\phi k_m) \right] + P_m - \rho \phi \varepsilon_m - D_m, \tag{54}$$

where

$$P_m = -\rho\phi\overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} : \nabla \langle \overline{\mathbf{u}} \rangle^i \qquad (55)$$

is the production rate of k_m due to the gradient of the macroscopic time-mean velocity $\langle \overline{\mathbf{u}} \rangle^i$.

Wang and Takle (1995), Antohe and Lage (1997), Getachewa *et al.* (2000) made use of the above equation for k_m considering for $\mathbf{R}'$ the Darcy-Forchheimer extended model with macroscopic time-fluctuation velocities $\langle \mathbf{u}' \rangle^i$. They have also neglected all dispersion terms that were grouped into D_m. Note also that the order of application of both volume- and time-average operators in this case cannot be changed. The quantity k_m is defined by applying first the volume operator to the fluctuating velocity field.

Transport equation for $\langle k \rangle^i$

The other procedure for composing the flow turbulent kinetic energy is to take the scalar product of eq. (47) by the microscopic fluctuating velocity $\mathbf{u}'$. Then apply both time- and volume-operators for obtaining an equation for $\langle k \rangle^i = \langle \overline{\mathbf{u}' \cdot \mathbf{u}'} \rangle^i / 2$. It is worth noting that in this case the order of application of both operations is immaterial since no additional mathematical operation (the scalar product) is conducted between the averaging processes. Therefore, this is the same as applying the volume operator to an equation for the microscopic k.

The volumetric average of a transport equation for k has been carried out in detail by Pedras and de Lemos (2001) and for only that the final resulting equation is presented, namely,

$$\rho\left[\frac{\partial}{\partial t}\left(\phi\langle k \rangle^i\right) + \nabla\cdot\left(\overline{\mathbf{u}}_D \langle k \rangle^i\right)\right] = \nabla\cdot\left[\left(\mu + \frac{\mu_{t_\phi}}{\sigma_k}\right)\nabla\left(\phi\langle k \rangle^i\right)\right] + P_i + G_i - \rho\phi\langle \varepsilon \rangle^i, \qquad (56)$$

where

$$P_i = -\rho\langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \nabla\overline{\mathbf{u}}_D, \qquad G_i = c_k \rho\phi \frac{\langle k \rangle^i |\overline{\mathbf{u}}_D|}{\sqrt{K}} \qquad (57)$$

are the production rate of $\langle k \rangle^i$ due to mean gradients of the seepage velocity $\overline{\mathbf{u}}_D$ and the generation rate of intrinsic k due the presence of the porous matrix, respectively. Also, in eq. (57) K is the medium permeability and c_k is a constant. As mentioned, eq. (56) has been proposed by Pedras and de Lemos (2001). Nevertheless, for the sake of completeness, a few steps of such a derivation are reproduced here. Application of the volume-average theorem to the transport equation for the turbulence kinetic energy k gives

$$\rho\left[\frac{\partial}{\partial t}\left(\phi\langle k\rangle^i\right)+\nabla\cdot\left(\phi\langle\overline{\mathbf{u}k}\rangle^i\right)\right]=\rho\nabla\cdot\left\{\phi\left\langle\overline{\mathbf{u}'\left(\frac{p'}{\rho}+k\right)}\right\rangle^i\right\}$$
$$+\,\mu\nabla^2(\phi\langle k\rangle^i)-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}:\nabla\overline{\mathbf{u}}\rangle^i-\rho\phi\langle\varepsilon\rangle^i\,, \qquad (58)$$

where the divergence of the right-hand side can be expanded as

$$\nabla\cdot(\phi\langle\overline{\mathbf{u}}\,k\rangle^i)=\nabla\cdot\left[\phi(\langle\overline{\mathbf{u}}\rangle^i\langle k\rangle^i+\langle{}^i\overline{\mathbf{u}}\,{}^ik\rangle^i)\right], \qquad (59)$$

where the first term is the convection of $\langle k\rangle^i$ due to the macroscopic velocity whereas the second is the convective transport due to spatial deviations of both k and $\mathbf{u}$. Likewise, the production term on the right of eq. (58) can be expanded as

$$-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}:\nabla\overline{\mathbf{u}}\rangle^i=-\rho\phi\left[\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i:\langle\nabla\overline{\mathbf{u}}\rangle^i+\langle{}^i(\overline{\mathbf{u}'\mathbf{u}'}):{}^i(\nabla\overline{\mathbf{u}})\rangle^i\right]. \qquad (60)$$

Similarly, the first term on the right of eq. (60) is the production of $\langle k\rangle^i$ due to the mean macroscopic flow and the second is the $\langle k\rangle^i$ production associated with spatial deviations of flow quantities k and $\mathbf{u}$.

The extra terms appearing in eqs. (59) and (60), respectively, represent extra transport/production of $\langle k\rangle^i$ due to the presence of solid material inside the integration volume. They should be null for the limiting case of clear fluid flow, or say, when $\phi\to 1 \Rightarrow K\to\infty$. Also, they should be proportional to the macroscopic velocity and to $\langle k\rangle^i$.

In paper by Pedras and de Lemos (2001), a proposal for those two extra transport/production rates of $\langle k\rangle^i$ was made as follows:

$$-\nabla\cdot\left(\phi\langle{}^i\overline{\mathbf{u}}k\rangle^i\right)-\rho\phi\left\langle{}^i(\overline{\mathbf{u}'\mathbf{u}'}):{}^i(\nabla\overline{\mathbf{u}})\rangle^i\right\rangle=G_i=c_k\rho\phi\frac{\langle k\rangle^i|\overline{\mathbf{u}}_D|}{\sqrt{K}}\,, \qquad (61)$$

where the constant c_k was numerically determined by fine flow computations considering the medium to be formed by circular rods, as well as longitudinal and transversal rods (Pedras and de Lemos 2003, de Lemos 2006). In spite of the variation in the medium morphology and the use of a wide range of porosity and Reynolds number, a value of 0.28 was found to be suitable for most calculations.

Comparison of transport equations for k_m and $\langle k\rangle^i$

A comparison between terms in the transport equation for k_m and $\langle k\rangle^i$ can now be conducted. Pedras and de Lemos (2000) has already shown the connection between these two quantities as being

$$\langle k\rangle^i=\frac{\overline{\langle\mathbf{u}'\cdot\mathbf{u}'\rangle^i}}{2}=\frac{\overline{\langle\mathbf{u}'\rangle^i\cdot\langle\mathbf{u}'\rangle^i}}{2}+\frac{\overline{\langle{}^i\mathbf{u}'\cdot{}^i\mathbf{u}'\rangle^i}}{2}=k_m+\frac{\overline{\langle{}^i\mathbf{u}'\cdot{}^i\mathbf{u}'\rangle^i}}{2}. \qquad (62)$$

Expanding the correlation forming the production term P_i by means of eq. (7), a connection between the two generation rates can also be written as follows:

$$P_i = -\rho \langle \overline{\mathbf{u}'\mathbf{u}'} \rangle^i : \nabla \overline{\mathbf{u}}_D = -\rho \left(\overline{\langle \mathbf{u}' \rangle^i \langle \mathbf{u}' \rangle^i} : \nabla \overline{\mathbf{u}}_D + \overline{\langle {}^i\mathbf{u}' \, {}^i\mathbf{u}' \rangle^i} : \nabla \overline{\mathbf{u}}_D \right)$$

$$= P_m - \rho \overline{\langle {}^i\mathbf{u}' \, {}^i\mathbf{u}' \rangle^i} : \nabla \overline{\mathbf{u}}_D \, . \tag{63}$$

We note that all the production rate of k_m, due to the mean flow, constitutes only part of the general production rate responsible for maintaining the overall level of $\langle k \rangle^i$.

6. MOVING POROUS BED

There are many situations of practical relevance where the porous substrate moves along the flow with a different velocity than that of the working fluid. Several manufacturing processes deal with such a configuration and the computations applied can be found in the literature (Hu Guoxin *et al*. 2003, Henda and Falcioni 2006). Biomass pelletization and preparation for energy production may also consider systems having a moving porous bed (Shimizu *et al*. 2006, Gobel *et al*. 2007). Therefore, the ability to realistically model such systems is of great advantage to a number of materials, food and energy production processes.

None of the above mentioned publications, however, has considered an extension of the mathematical model detailed in Pedras and de Lomos (2000, 2001, 2003). The purpose of this section is to propose a new model for treating moving porous beds based on the developments fully documented in de Lemos (2006).

Definitions and hypotheses

First, it is important to emphasize that only cases where the solid phase velocity is kept constant will be considered here. Not considered also is the case where the gaps in the time series, at points intermittently occupied by different phases, requires alternative averaging procedures (e.g., moving grains). For a broader discussion on such cases, see Nikora *et al*. (2007).

The situation investigated here corresponds to a moving bed crossing a fixed volume in addition to a flowing fluid, which is not necessarily moving with a velocity aligned with the solid phase velocity (see Fig. 1). The steps below show some basic definitions prior to presenting a proposal for a set of transport equations for analyzing moving bed systems.

A general form for a volume-average of any property φ, distributed within phase γ that occupy volume ΔV_γ, can be written as (Gray and Lee 1977)

$$\langle \varphi \rangle^\gamma = \frac{1}{\Delta V_\gamma} \int_{\Delta V_\gamma} \varphi \, \mathrm{d}V_\gamma \, . \tag{64}$$

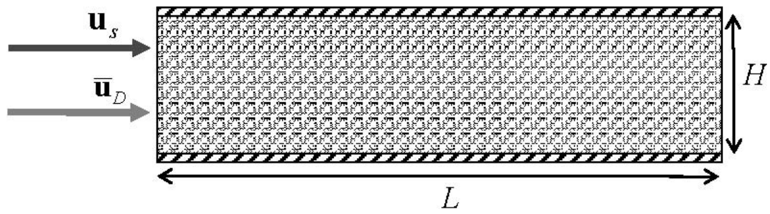

Fig. 1. Two-dimensional channel with a moving porous bed.

The volume ratio occupied by phase γ will be $\phi^\gamma = \Delta V_\gamma/\Delta V$. If there are two phases, a solid ($\gamma = s$) and a fluid phase ($\gamma = f$), volume average can be established on both regions. Also,

$$\phi^s = \Delta V_s / \Delta V = 1 - \Delta V_f / \Delta V = 1 - \phi^f \tag{65}$$

and for simplicity of notation one can drop the superscript f to get $\phi^s = 1 - \phi$.

As such, calling the instantaneous local velocities for the solid and fluid phases, $\mathbf{u}_s$ and $\mathbf{u}$, respectively, one can obtain the average for the solid velocity, within the solid phase, as follows:

$$\langle \mathbf{u} \rangle^s = \frac{1}{\Delta V_s} \int_{\Delta V_s} \mathbf{u}_s \, \mathrm{d} V_s \tag{66}$$

which, in turn, can be related to an average velocity related to the entire REV as

$$\mathbf{u}_S = \overbrace{\frac{\Delta V_s}{\Delta V}}^{(1-\phi)} \underbrace{\frac{1}{\Delta V_s} \int_{\Delta V_s} \mathbf{u}_s \, \mathrm{d} V_s}_{\langle \mathbf{u} \rangle^s} . \tag{67}$$

A further approximation herein is that the porous bed is rigid and moves with a steady average velocity $\mathbf{u}_S$. Note that the condition of steady velocity for the solid phase implies $\mathbf{u}_S = \overline{\mathbf{u}}_S = \text{const}$, where the overbar denotes, as before, time-averaging.

For the fluid phase, the intrinsic (fluid) volume average gives, after using the subscript i also for consistency with the above,

$$\langle \overline{\mathbf{u}} \rangle^i = \frac{1}{\Delta V_f} \int_{\Delta V_f} \overline{\mathbf{u}} \, \mathrm{d} V_f . \tag{68}$$

Both velocities can then be written as

$$\overline{\mathbf{u}}_D = \phi \langle \overline{\mathbf{u}} \rangle^i, \qquad \mathbf{u}_S = (1-\phi) \langle \mathbf{u} \rangle^s = \text{const} . \tag{69}$$

A total-volume based relative velocity is then defined as

$$\overline{\mathbf{u}}_{rel} = \overline{\mathbf{u}}_D - \mathbf{u}_S . \tag{70}$$

Modelled transport equations

A macroscopic time-mean Navier-Stokes (NS) equation for an incompressible fluid with constant properties can be modelled as (see, Pedras and de Lemos 2001, for details)

$$\rho\nabla\cdot\left(\frac{\bar{\mathbf{u}}_D\bar{\mathbf{u}}_D}{\phi}\right)=-\nabla\left(\phi\langle\bar{p}\rangle^i\right)+\mu\nabla^2\bar{\mathbf{u}}_D+\nabla\cdot\left(-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i\right)-\left[\frac{\mu\phi}{K}\bar{\mathbf{u}}_D+\frac{c_F\phi\rho|\bar{\mathbf{u}}_D|\bar{\mathbf{u}}_D}{\sqrt{K}}\right], \tag{71}$$

where the last two terms in eq. (71) are known as the Darcy and the Forchheimer drags, respectively. These terms represent the viscous and net pressure forces felt by the fluid after passing through the porous bed, whose analytical form is presented by eq. (37). Also, in eq. (71) K is the medium permeability, c_F is the Forchheimer coefficient and $-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i$ is the Macroscopic Reynolds Stress Tensor (MRST) modelled as

$$-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i=\mu_{t_\phi}2\langle\bar{\mathbf{D}}\rangle^v-\frac{2}{3}\phi\rho\langle k\rangle^i\mathbf{I}. \tag{72}$$

Further,

$$\langle\bar{\mathbf{D}}\rangle^v=\frac{1}{2}\left[\nabla\left(\phi\langle\bar{\mathbf{u}}\rangle^i\right)+\left[\nabla\left(\phi\langle\bar{\mathbf{u}}\rangle^i\right)\right]^T\right] \tag{73}$$

is the macroscopic deformation tensor, $\langle k\rangle^i$ is the intrinsic average for k, as above, and μ_{t_ϕ} is the macroscopic turbulent viscosity, which is modelled here similarly to the case of clear fluid flow. As such, a proposal for μ_{t_ϕ} was presented in Pedras and de Lemos (2001) as

$$\mu_{t_\phi}=\rho c_\mu\langle k\rangle^{i^2}\Big/\langle\varepsilon\rangle^i. \tag{74}$$

For a fixed bed, the final form of eq. (71) reads, after incorporating the models given by eqs. (72)-(74),

$$\rho\left[\nabla\cdot\left(\frac{\bar{\mathbf{u}}_D\bar{\mathbf{u}}_D}{\phi}\right)\right]-\nabla\cdot\left\{(\mu+\mu_{t_\phi})\left[\nabla\bar{\mathbf{u}}_D+(\nabla\bar{\mathbf{u}}_D)^T\right]\right\}$$
$$=-\nabla(\phi\langle\bar{p}\rangle^i)-\left[\frac{\mu\phi}{K}\bar{\mathbf{u}}_D+\frac{c_F\phi\rho|\bar{\mathbf{u}}_D|\bar{\mathbf{u}}_D}{\sqrt{K}}\right]. \tag{75}$$

For the configuration shown in Fig. 1 and assuming that the relative movement between the two phases is described by eq. (70), the momentum equation reads

$$\rho\left[\nabla\cdot\left(\frac{\bar{\mathbf{u}}_D\bar{\mathbf{u}}_D}{\phi}\right)\right]-\nabla\cdot\left\{(\mu+\mu_{t_\phi})\left[\nabla\bar{\mathbf{u}}_D+(\nabla\bar{\mathbf{u}}_D)^T\right]\right\}$$
$$=-\nabla(\phi\langle\bar{p}\rangle^i)-\left[\frac{\mu\phi}{K}\bar{\mathbf{u}}_{rel}+\frac{c_F\phi\rho|\bar{\mathbf{u}}_{rel}|\bar{\mathbf{u}}_{rel}}{\sqrt{K}}\right]. \tag{76}$$

Strictly speaking, eq. (76) should be valid for $\mathbf{u}_S/\overline{\mathbf{u}}_D > 0$. A corresponding transport equation for $\langle k\rangle^i$ can be written as

$$\rho\left[\nabla\cdot(\overline{\mathbf{u}}_D\langle k\rangle^i)\right]=\nabla\cdot\left[\left(\mu+\frac{\mu_{t_\phi}}{\sigma_k}\right)\nabla(\phi\langle k\rangle^i)\right]$$

$$-\rho\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i:\nabla\overline{\mathbf{u}}_D+c_k\rho\frac{\phi\langle k\rangle^i\left|\overline{\mathbf{u}}_{rel}\right|}{\sqrt{K}}-\rho\phi\langle\varepsilon\rangle^i\,, \tag{77}$$

where the generation rate due to the porous substrate, G^i, which was included in eq. (56), now depends on $\overline{\mathbf{u}}_{rel}$.

7. CONCLUSIONS

In this paper, a new methodology for the analysis of turbulent flow in permeable media was detailed. A novel concept, called the double-decomposition idea, was reviewed to show how a variable can be decomposed in both time and volume in order to simultaneously account for deviations in space and fluctuations in time around mean values. Exact transport equations for the mean and turbulent flow fields were presented. An extension to the case of a moving porous bed is also presented.

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