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ON THE DEFINITION OF TURBULENT KINETIC ENERGY FOR FLOW IN POROUS MEDIA

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ABSTRACT

In the literature, there are two distinct approaches for developing turbulent models for flow in a porous medium. The first one starts with the macroscopic equations using the extended Darcy-Forchheimer model. The second method considers first the microscopic balance equations. In both cases, time and volume averaging operators are applied in a different order. The turbulence kinetic energy equation resulting from application of the two averaging operators, following both orders of integration, are different. In this work, a new double-decomposition (time and volume) methodology is suggested and the differences between those two mathematical treatments are highlighted.

Introduction

Engineering systems applied to enhanced oil reservoir recovery, fluidized bed combustion, underground spreading of chemical waste, combustion in an inert porous matrix and chemical catalytic reactors are examples of applications of the study of flow through porous media.

Based on the pore Reynolds number, Re_p, different flow regimes are recognized in the literature. For Re_p < 150~200, classical mathematical treatment of flow in porous media invokes the notion of a Representative Elementary Volume (REV) for which balance equations governing momentum, energy and mass transfer are written [1-7]. However, the mathematical description of fully turbulent flow regime (Re_p > 300) has given rise to interesting discussions and remains a controversial issue. For this Re_p range, turbulence models presented in the literature follow two contradictory approaches. In the first one [8-10], governing equations for the mean and turbulent fields are obtained by time-averaging the macroscopic equations. In the second method [11-14], a volume-average operator is applied to the local time-averaged equation. Or say, in the first case, volume-average is taken first and then time averaging is applied. In the latter method, the order of averaging is reversed. In the literature, these two different approaches lead to different governing equations.

Motivated by the foregoing, the objective of this work is to show that, for a rigid saturated medium, the two approaches lead to similar equations for the mean flow. However, the turbulence kinetic energy resulting from application of the two averaging operators, following both orders of integration, are different. The connection between these two quantities is here highlighted.

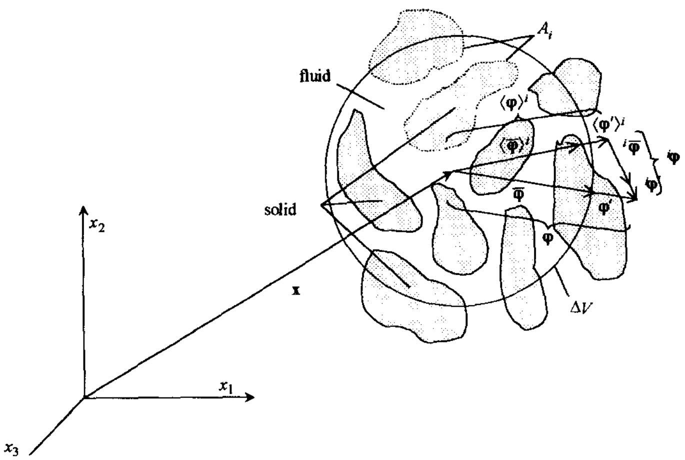

FIG.1
Representative elementary volume; space and time fluctuations.

The Averaging Operators

Local volume-average. The macroscopic governing equation for flow through a porous substratum can be obtained by volume averaging the corresponding microscopic equations over a Representative Elementary Volume, ΔV , schematically shown in Fig. 1. For a general fluid property, φ , the intrinsic and volumetric averages are related through the porosity ϕ as,

$$\langle\varphi\rangle^i = \frac{1}{\Delta V_f}\int_{\Delta V_f}\varphi dV;\ \langle\varphi\rangle^v = \phi\langle\varphi\rangle^i;\ \phi = \frac{\Delta V_f}{\Delta V} \tag{1}$$

where ΔV_f is the volume of the fluid contained in ΔV . Fig. 1 illustrates the idea that any property φ can be defined as the sum of $\langle\varphi\rangle^i$ and a term related to its spatial variation within the REV, ${}^i\varphi$, such that

$$\varphi = \langle\varphi\rangle^i + {}^i\varphi \tag{2}$$

The spatial deviation is the difference between the local value (microscopic) and its intrinsic (fluid based average) value [5]. Further, from (1) and (2) one gets $\langle {}^i\varphi\rangle^i = 0$.

For deriving the flow governing equations, it is necessary to know the relationship between the volumetric average of derivatives and the derivatives of the volumetric average. These relationships are presented in references [5,15,16] and are known as the Theorem of Local Volumetric Average. They are written as,

$$\langle \nabla \varphi \rangle^v = \nabla(\phi \langle \varphi \rangle^i) + \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\varphi dS$$
$$\langle \nabla \cdot \boldsymbol{\varphi} \rangle^v = \nabla \cdot (\phi \langle \boldsymbol{\varphi} \rangle^i) + \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\cdot\boldsymbol{\varphi} dS \qquad (3)$$
$$\langle \frac{\partial \varphi}{\partial t} \rangle^v = \frac{\partial}{\partial t}(\phi \langle \varphi \rangle^i) - \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\cdot(\mathbf{u}_i \varphi) dS$$

where A_i and $\mathbf{u}_i$ are the interfacial area and velocity of phase f and $\mathbf{n}$ is the unity vector normal to A_i. For single-phase flow, phase f is the fluid itself and $\mathbf{u}_i = 0$ if the porous substrate is assumed to be fixed. In developing equations (3) the only restriction applied was the independence of ΔV in relation to time and space. If the medium is further assumed to be rigid then ΔV_f is dependent on space but is constant with time [15].

Time-average. The need for considering time fluctuations occurs when turbulence effects are of concern. The microscopic time-averaged equations are obtained from the instantaneous microscopic equations. For that, the time-average value of property φ associated with the fluid is given as:

$$\overline{\varphi} = \frac{1}{\Delta t}\int_t^{t+\Delta t} \varphi dt \qquad (4)$$

where Δt is the integration time interval. The instantaneous property φ can be defined as the sum of the time average value, $\overline{\varphi}$, plus the fluctuating component, φ', in the form,

$$\varphi = \overline{\varphi} + \varphi' \qquad (5)$$

with $\overline{\varphi'} = 0$.

Commutative Properties. From the definition of volume average (1) and time average (4), one can conclude that the time average of the volume average of property φ is given by:

$$\overline{\langle \varphi \rangle^v} = \frac{1}{\Delta t}\int_t^{t+\Delta t} \left[\frac{1}{\Delta V}\int_{\Delta V_f} \varphi dV \right] dt \qquad (6)$$

On the other hand, the volume average of the time average is,

$$\langle \overline{\varphi} \rangle^v = \frac{1}{\Delta V} \int_{\Delta V_f} \left[\frac{1}{\Delta t} \int_t^{t+\Delta t} \varphi dt \right] dV \tag{7}$$

As mentioned, for a rigid medium, the volume of fluid, ΔV_f, will be dependent only on space but not on time. If the time interval chosen for temporal averaging, Δt, is also the same for all REV, then the volumetric average commutes with time average because both integration domains in (6)-(7) are independent of each other. In this case, the order of application of average operators is immaterial so that equations (6) and (7) will lead to:

$$\overline{\langle \varphi \rangle^v} = \langle \overline{\varphi} \rangle^v \text{ or } \overline{\langle \varphi \rangle^i} = \langle \overline{\varphi} \rangle^i \tag{8}$$

Double decomposition – Space and time fluctuations. From definitions (1) and (5) one gets,

$$\langle \varphi \rangle^i = \frac{1}{\Delta V_f} \int_{\Delta V_f} \varphi dV = \frac{1}{\Delta V_f} \int_{\Delta V_f} (\overline{\varphi} + \varphi') dV = \langle \overline{\varphi} \rangle^i + \langle \varphi' \rangle^i \tag{9}$$

and combining equations (2) and (4) one arrives at,

$$\overline{\varphi} = \frac{1}{\Delta t} \int_t^{t+\Delta t} \varphi dt = \frac{1}{\Delta t} \int_t^{t+\Delta t} (\langle \varphi \rangle^i + {}^i\varphi) dt = \overline{\langle \varphi \rangle^i} + \overline{{}^i\varphi} \tag{10}$$

Further, the space-averaged value $\langle \varphi \rangle^i$ can also be decomposed into a time mean and fluctuating component as:

$$\langle \varphi \rangle^i = \overline{\langle \varphi \rangle^i} + \langle \varphi \rangle^{i\prime}, \tag{11}$$

Using now the fact that the averages commute (equation (8)), a comparison of equations (9) and (11) validates the following relationship:

$$\langle \varphi' \rangle^i = \langle \varphi \rangle^{i\prime} \tag{12}$$

Similarly, if one considers the time average component having also a spatial distribution, one has,

$$\overline{\varphi} = \langle \overline{\varphi} \rangle^i + {}^i\overline{\varphi}\,. \tag{13}$$

Likewise, a comparison between (10) and (13), in light of (8), gives,

$${}^i\overline{\varphi} = \overline{{}^i\varphi} \tag{14}$$

Further, since both time and space decompositions are based on the same value for φ, one can promptly write,

$$\varphi = \langle\varphi\rangle^i + {}^i\varphi = \overline{\varphi} + \varphi' \tag{15}$$

Applying now a full double decomposition to all terms in (15) and using (8)-(12)-(14), one finally gets for φ,

$$\varphi = \langle\overline{\varphi}\rangle^i + \langle\varphi'\rangle^i + {}^i\overline{\varphi} + {}^i\varphi' = \overline{\langle\varphi\rangle^i} + \langle\varphi\rangle^{i'} + \overline{{}^i\varphi} + {}^i\varphi' \tag{16}$$

The time average of the fluctuating component $\varphi' = \langle\varphi\rangle^{i'} + {}^i\varphi'$ gives further $\overline{{}^i\varphi'} = 0$. Likewise, volume averaging the spatial component ${}^i\varphi = {}^i\overline{\varphi} + {}^i\varphi'$ will result in $\langle{}^i\varphi'\rangle^i = 0$.

With these ideas in mind, integration of local (microscopic) momentum equation applied to the domain shown in Fig. 1 can be more easily treated. In addition, one can show that the order of integration (space and time) of this equation is, in fact, immaterial.

Momentum Equation

The development to follow assumes single-phase flow in a saturated, rigid porous medium (ΔV_f independent of time) for which, in accordance with (8), time average operation on variable φ commutes with space average. Application of the double decomposition idea in equation (16) to the convection term in the momentum equation lead to four different terms. Not all of these terms are considered in the same analysis in the literature.

One average operator. The microscopic momentum equation for a fluid with constant properties is given by the Navier-Stokes equation as

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot(\mathbf{u}\mathbf{u})\right] = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{g} \tag{17}$$

Its time-average using $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ gives

$$\rho\left[\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}})\right] = -\nabla\overline{p} + \mu\nabla^2\overline{\mathbf{u}} + \nabla\cdot(-\rho\overline{\mathbf{u}'\mathbf{u}'}) + \rho\mathbf{g} \tag{18}$$

where the stresses, $-\rho\overline{\mathbf{u}'\mathbf{u}'}$, are the well-known Reynolds stresses. On the other hand, the volumetric average of (17) using the Theorem of Local Volumetric Average (equation (3)), results in

$$\rho\left[\frac{\partial}{\partial t}(\phi\langle\mathbf{u}\rangle^i)+\nabla\cdot[\phi\langle\mathbf{uu}\rangle^i]\right]=-\nabla(\phi\langle p\rangle^i)+\mu\nabla^2(\phi\langle\mathbf{u}\rangle^i)+\phi\rho\mathbf{g}+\mathbf{R} \tag{19}$$

where $\mathbf{R}=\frac{\mu}{\Delta V}\int_{A_i}\mathbf{n}\cdot(\nabla\mathbf{u})dS-\frac{1}{\Delta V}\int_{A_i}\mathbf{n}p\,dS$ represents the total drag force per unit volume due to the presence of the porous matrix, being composed by both viscous drag and form (pressure) drag. Further, using (2) to write $\mathbf{u}=\langle\mathbf{u}\rangle^i+{}^i\mathbf{u}$ in the convection term, one gets,

$$\rho\left[\frac{\partial}{\partial t}(\phi\langle\mathbf{u}\rangle^i)+\nabla\cdot[\phi\langle\mathbf{u}\rangle^i\langle\mathbf{u}\rangle^i]\right]=-\nabla(\phi\langle p\rangle^i)+\mu\nabla^2(\phi\langle\mathbf{u}\rangle^i)-\nabla\cdot[\phi\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i]+\phi\rho\mathbf{g}+\mathbf{R} \tag{20}$$

Hsu & Cheng, 1990 [17], points that the third term on the right of (20), $\nabla\cdot(\phi\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i)$, represents the hydrodynamic dispersion due to spatial deviations. Note that equation (20) models typical porous media flow for Re_p<150-200. When extending the analysis to turbulent flow, time varying quantities have to be considered.

Two average operators. The set of equations (18) and (20) are used when treating turbulent flow in clear fluid or low Re_p porous media flow, respectively. In each one of those equations only one averaging operator was applied, either time or volume, respectively. In this work, an investigation on the use of both operators in conducted with the objective of modeling turbulent flow in porous media.

The volume average of (18) gives for the time mean flow in a porous medium,

$$\rho\left[\frac{\partial}{\partial t}(\phi\langle\overline{\mathbf{u}}\rangle^i)+\nabla\cdot(\phi\langle\overline{\mathbf{u}}\,\overline{\mathbf{u}}\rangle^i)\right]=-\nabla(\phi\langle\overline{p}\rangle^i)+\mu\nabla^2(\phi\langle\overline{\mathbf{u}}\rangle^i)+\nabla\cdot(-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i)+\phi\rho\mathbf{g}+\overline{\mathbf{R}} \tag{21}$$

where $\overline{\mathbf{R}}=\frac{\mu}{\Delta V}\int_{A_i}\mathbf{n}\cdot(\nabla\overline{\mathbf{u}})dS-\frac{1}{\Delta V}\int_{A_i}\mathbf{n}\overline{p}\,dS$. Likewise, applying the time average operation to (19), one gets:

$$\begin{aligned}\rho&\left[\frac{\partial}{\partial t}\overline{(\phi\langle\overline{\mathbf{u}}+\mathbf{u}'\rangle^i)}+\nabla\cdot\overline{(\phi\langle(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')\rangle^i)}\right]\\&=-\nabla\overline{(\phi\langle\overline{p}+p'\rangle^i)}+\mu\nabla^2\overline{(\phi\langle\overline{\mathbf{u}}+\mathbf{u}'\rangle^i)}+\phi\rho\mathbf{g}+\overline{\mathbf{R}},\end{aligned} \tag{22}$$

Dropping terms containing only one time fluctuating quantity results in,

$$\rho\left[\frac{\partial}{\partial t}(\phi\langle\overline{\mathbf{u}}\rangle^i)+\nabla\cdot(\phi\langle\overline{\mathbf{u}}\,\overline{\mathbf{u}}\rangle^i)\right]=-\nabla(\phi\langle\overline{p}\rangle^i)+\mu\nabla^2(\phi\langle\overline{\mathbf{u}}\rangle^i)+\nabla\cdot(-\rho\phi\langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i)+\phi\rho\mathbf{g}+\overline{\mathbf{R}} \tag{23}$$

where $\overline{\mathbf{R}} = \frac{\mu}{\Delta V}\int_{A_i} \mathbf{n}\cdot[\nabla\overline{(\overline{\mathbf{u}}+\mathbf{u}')}]dS - \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\overline{(\overline{p}+p')}dS = \frac{\mu}{\Delta V}\int_{A_i} \mathbf{n}\cdot(\nabla\overline{\mathbf{u}})dS - \frac{1}{\Delta V}\int_{A_i} \mathbf{n}\overline{p}dS$.

Comparing (21) and (23) one can see that for the momentum equation the order of the application of both averaging operators is immaterial. The source of controversies comes from the convection term, as seen below. Also, it will be shown that for the turbulent kinetic energy, the order of application of the average operators will lead to different equations.

Convection Term -Double decomposition: When the double decomposition idea (see equation (16)) is applied to the convection term of (17), four different terms are obtained. Starting with time decomposition and applying both average operators gives,

$$\nabla\cdot\overline{(\phi\langle\mathbf{uu}\rangle^i)} = \nabla\cdot\overline{(\phi\langle(\overline{\mathbf{u}}+\mathbf{u}')(\overline{\mathbf{u}}+\mathbf{u}')\rangle^i)} = \nabla\cdot[\phi(\langle\overline{\mathbf{u}}\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i)] \tag{24}$$

Using equation (13) to write $\overline{\mathbf{u}} = \langle\overline{\mathbf{u}}\rangle^i + {}^i\overline{\mathbf{u}}$ and plugging it into (24),

$$\begin{aligned}&\nabla\cdot[\phi(\langle\overline{\mathbf{u}}\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i)]\\ &= \nabla\cdot\{\phi[\langle(\langle\overline{\mathbf{u}}\rangle^i + {}^i\overline{\mathbf{u}})(\langle\overline{\mathbf{u}}\rangle^i + {}^i\overline{\mathbf{u}})\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i]\} = \nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i]\}.\end{aligned} \tag{25}$$

Now, writing $\mathbf{u}' = \langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}'$ and substituting it into (25) gives,

$$\begin{aligned}&\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \langle\overline{\mathbf{u}'\mathbf{u}'}\rangle^i]\} =\\ &\quad\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle(\langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}')(\langle\mathbf{u}'\rangle^i + {}^i\mathbf{u}')\rangle^i}]\} =\\ &\quad\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle(\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i + 2\langle\mathbf{u}'\rangle^i\,{}^i\mathbf{u}' + {}^i\mathbf{u}'\,{}^i\mathbf{u}')\rangle^i}]\} =\\ &\quad\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + 2\overline{\langle\langle\mathbf{u}'\rangle^i\,{}^i\mathbf{u}'\rangle^i} + \langle\overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i]\},\end{aligned} \tag{26}$$

The fourth term on the right of (26) contains only one space varying quantity and will vanish under the application of volume integration. Equation (26) will then be reduced to,

$$\nabla\cdot\overline{(\phi\langle\mathbf{uu}\rangle^i)} = \nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i + \overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i} + \langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i + \langle\overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i]\} \tag{27}$$

Another route to follow for reaching the same results is to start out with the application of the space decomposition in the convection term, as usually done in classical mathematical treatment of porous media flow. Then one has [17],

$$\nabla\cdot\overline{(\phi\langle\mathbf{uu}\rangle^i)} = \nabla\cdot\overline{[\phi\langle(\langle\mathbf{u}\rangle^i + {}^i\mathbf{u})(\langle\mathbf{u}\rangle^i + {}^i\mathbf{u})\rangle^i]} = \nabla\cdot\overline{[\phi(\langle\mathbf{u}\rangle^i\langle\mathbf{u}\rangle^i + \langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i)]} \tag{28}$$

The time average of the RHS of (28), using equation (9) to express $\langle\mathbf{u}\rangle^i = \langle\overline{\mathbf{u}}\rangle^i + \langle\mathbf{u}'\rangle^i$, becomes,

$$
\begin{aligned}
&\nabla\cdot\overline{[\phi(\langle\mathbf{u}\rangle^i\langle\mathbf{u}\rangle^i+\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i)]}\\
&=\nabla\cdot\overline{\{\phi[(\langle\overline{\mathbf{u}}\rangle^i+\langle\mathbf{u}'\rangle^i)(\langle\overline{\mathbf{u}}\rangle^i+\langle\mathbf{u}'\rangle^i)+\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i]\}}=\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i+\overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}+\overline{\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i}]\},
\end{aligned}
\tag{29}
$$

Inserting ${}^i\mathbf{u}={}^i\overline{\mathbf{u}}+{}^i\mathbf{u}'$ into (29) gives,

$$
\begin{aligned}
&\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i+\overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}+\overline{\langle{}^i\mathbf{u}\,{}^i\mathbf{u}\rangle^i}]\}\\
&\quad=\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i+\overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}+\overline{\langle({}^i\overline{\mathbf{u}}+{}^i\mathbf{u}')({}^i\overline{\mathbf{u}}+{}^i\mathbf{u}')\rangle^i}]\}\\
&\quad=\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i+\overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}+\langle\overline{{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}}+2\overline{{}^i\overline{\mathbf{u}}\,{}^i\mathbf{u}'}+\overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i]\}.
\end{aligned}
\tag{30}
$$

Application of the time average operator to the forth term of (30), containing only one fluctuating component, vanishes it, and equation (30) becomes,

$$
\nabla\cdot\overline{(\phi\langle\mathbf{u}\mathbf{u}\rangle^i)}=\nabla\cdot\{\phi[\underbrace{\langle\overline{\mathbf{u}}\rangle^i\langle\overline{\mathbf{u}}\rangle^i}_{\mathrm{I}}+\underbrace{\overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}}_{\mathrm{II}}+\underbrace{\langle{}^i\overline{\mathbf{u}}\,{}^i\overline{\mathbf{u}}\rangle^i}_{\mathrm{III}}+\underbrace{\langle\overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i}_{\mathrm{IV}}]\}
\tag{31}
$$

which is the same result of (27). A physical significance of all four terms on the right of (31) can be discussed as: I - Convective term of macroscopic mean velocity, II - Turbulent (Reynolds) stresses divided by density ρ due to the fluctuating component of the macroscopic velocity, III - Dispersion associated with spatial fluctuations of microscopic time mean velocity. Note that this term is also present in laminar flow, or say, when $Re_p < 150$-200, and IV - Turbulent dispersion in a porous medium due to both time and spatial fluctuations of the microscopic velocity.

Turbulent Kinetic Energy

In the section above, it was shown that the order of application of both time and volume average operators is immaterial regarding the final momentum equation obtained. Although not shown here, this is also true for the continuity and energy equations. For clear fluids ($\phi = 1$), the turbulent kinetic energy defined as $k=\overline{\mathbf{u}\cdot\mathbf{u}}/2$ is used by most turbulence models. However, how to determine k in a porous medium is still an open question. Depending on the order of application of the average operators, the final governing equation for the flow turbulent kinetic energy will refer to different quantities. To show that is the purpose of this section.

The starting point for deriving a transport equation for k is the microscopic velocity fluctuation $\mathbf{u}'$. An equation for $\mathbf{u}'$ can be written after subtracting (18) from (17), resulting in [18]:

$$
\rho\left\{\frac{\partial\mathbf{u}'}{\partial t}+\nabla\cdot[\overline{\mathbf{u}}\mathbf{u}'+\mathbf{u}'\overline{\mathbf{u}}+\mathbf{u}'\mathbf{u}'-\overline{\mathbf{u}'\mathbf{u}'}]\right\}=-\nabla p'+\mu\nabla^2\mathbf{u}'.
\tag{32}
$$

Now, the volumetric average of (32) using the Theorem of Local Volumetric Average (equation (3)) will give,

$$\rho\frac{\partial}{\partial t}(\phi\langle\mathbf{u}'\rangle^i)+\rho\nabla\cdot\{\phi[\langle\overline{\mathbf{u}}\rangle^i\langle\mathbf{u}'\rangle^i+\langle\mathbf{u}'\rangle^i\langle\overline{\mathbf{u}}\rangle^i+\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i+ \\ \langle{}^i\overline{\mathbf{u}}\,{}^i\mathbf{u}'\rangle^i+\langle{}^i\mathbf{u}'\,{}^i\overline{\mathbf{u}}\rangle^i+\langle{}^i\mathbf{u}'\,{}^i\mathbf{u}'\rangle^i-\overline{\langle\mathbf{u}'\rangle^i\langle\mathbf{u}'\rangle^i}-\langle\overline{{}^i\mathbf{u}'\,{}^i\mathbf{u}'}\rangle^i]\}= \\ -\nabla(\phi\langle p'\rangle^i)+\mu\nabla^2(\phi\langle\mathbf{u}'\rangle^i)+\mathbf{R}-\overline{\mathbf{R}}, \tag{33}$$

where $\mathbf{R}-\overline{\mathbf{R}}=\frac{\mu}{\Delta V}\int_{A_i}\mathbf{n}\cdot(\nabla\mathbf{u}')dS-\frac{1}{\Delta V}\int_{A_i}\mathbf{n}p'dS$.

Equation (33) can also be obtained by subtracting the macroscopic mean velocity equation (21) from the instantaneous macroscopic velocity equation (19).

From this point on there are two distinct approaches to determine a transport equation associated with the flow turbulence kinetic energy. References [8,9,10] based their turbulence models on $k_m=\overline{\langle\mathbf{u}'\rangle^i\cdot\langle\mathbf{u}'\rangle^i}/2$. They started with equation (33), took the scalar product of it by $\langle\mathbf{u}'\rangle^i$ and applied the time-average operator. They have therefore used the volume operator **first**, and then the time averaging was applied. On the other hand, if one starts out with equation (32) and one takes the scalar product of it by $\mathbf{u}'$ before time-averaging, one ends up, after volume averaging, with an equation for $\langle k\rangle^i=\langle\overline{\mathbf{u}'\cdot\mathbf{u}'}\rangle^i/2$. This was the path followed in references [11-14]. Now, using the double decomposition idea suggested herein, one can clarify the connection between these two quantities as being,

$$\langle k\rangle^i=\langle\overline{\mathbf{u}'\cdot\mathbf{u}'}\rangle^i/2=\overline{\langle\mathbf{u}'\rangle^i\cdot\langle\mathbf{u}'\rangle^i}/2+\langle\overline{{}^i\mathbf{u}'\cdot{}^i\mathbf{u}'}\rangle^i/2 \\ =k_m+\langle\overline{{}^i\mathbf{u}'\cdot{}^i\mathbf{u}'}\rangle^i/2 \tag{34}$$

The last term on the right of (34) is the extra turbulent kinetic energy obtained by adding up elements of the main diagonal of term (IV) in equation (31). As seen, models based on k_m do not fully account for all of the turbulent kinetic energy associated with the flow. The effect of neglecting this contribution for accurate analysis of turbulent flows in porous media stimulates further work on this subject [19].

Conclusions

Both average operators for time and volume commute, as long as the domains of integration are independent of each other. Therefore, when obtaining the macroscopic governing equation for momentum (and also for continuity and energy, not shown here) the order of application of the operators is immaterial and the equations end up having the same form. The restrictions applied are that the porous medium must be undeformable and saturated by a single-phase fluid.

When obtaining macroscopic transport equation for the turbulent kinetic energy, however, the order of application of averages will imply in a different quantity being transported. This is because there is an additional mathematical operation needed for forming the turbulent kinetic energy. This operation consists in the scalar product of the fluctuating velocity by its own transport equation. When this scalar product is taken after the volume integration process [8-10], the quantity undergoing time integration is $(\langle \mathbf{u}' \rangle^i \cdot \langle \mathbf{u}' \rangle^i)$. On the other hand, when proceeding with the scalar product first at the microscopic level, a different variable is subjected to time integration ($\mathbf{u}' \cdot \mathbf{u}'$) [11-14]. In the second method, a more complete form of the turbulence kinetic energy is obtained and all microscopic effects as considered.

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